Wave packets in the Schwartz space of a reductive p-adic symmetric space

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Abstract We form wave packets in the Schwartz space of a reductive p-adic symmetric space for certain famillies of tempered functions. We show how to construct such families from Eisenstein integrals.

1 Introduction

Let G be the group of $\mathbf{F}$-points of an algebraic group, $\underline{G}$, defined over $\mathbf{F}$, where $\mathbf{F}$ is a nonarchimedean local field of characteristic different from 2. Let H be the group of $\mathbf{F}$-points of an open $\mathbf{F}$-subgroup of the fixed point group of an involution of $\underline{G}$ defined over $\mathbf{F}$.

We introduce the space $\mathcal{A}_{temp}(H\backslash G)$ of smooth tempered functions on $H\backslash G$. They are the tempered functions which are generalized coefficients of an H-fixed linear form ξ on an admissible smooth G-module V, when V and ξ varies.

Using the theory of the constant term (cf. [L], [KT1]), we introduce the weak constant term of elements of $\mathcal{A}_{temp}(H\backslash G)$ as it was made in [W] for tempered functions on the group.

Then we introduce families of elements of $\mathcal{A}_{temp}(H\backslash G)$ of type I, by conditions on their weak constant term. Then the conditions are strengthened to introduce families of type I', II'. This is the analog of the families used in [BaCD] for the real case.
Important examples of such families are given (cf. Theorem 2) in terms of Eisenstein integrals, due to the main results of [CD].

Then, following closely [W], we show that one can form wave packets in the Schwartz space for such families (Theorem 1) like in [BaCD] for the real case. Notice also that the intermediate Proposition 2 is the analogous of the important Lemma 7.1 of [A].

This result has two interests. First, the recent work of Sakerallidis and Venkatesh [SaV] on spherical varieties includes in particular the L^2-Plancherel formula for $H\backslash G$, when G is split and the characteristic of $\mathbf{F}$ is equal to zero. It should be possible using our result to determine the Fourier transform of the Schwartz space for these symmetric spaces. This should be entirely analog to the work [DO] for affine Hecke algebras.

The second interest is that, when the Cartan decomposition is explicit and the corresponding integral formula is known as in [O], our results might allow to use the technique of truncation as in [D1], [D2] for the real case, to get the Plancherel formula and the Fourier transform of the Schwartz space. This should give examples not treated by Sakerallidis and Venkatesh.

In order to prepare the truncation process we prove (cf. Propostion 3) a canonical decomposition $\mathcal{A}_{temp}(H\backslash G) = \mathcal{A}_2(H\backslash G) \oplus \mathcal{A}_{temp,c}(H\backslash G)$. If G is semisimple, this decomposition is essentially the direct sum of the space of the square integrable elements and its orthogonal in a suitable sense, as there is no natural scalar product on $\mathcal{A}_{temp}(H\backslash G)$. In general, we manage to reduce to this case. The square integrability criterion of [KT2] is very useful for the proof of the decomposition. To our knowledge, this decomposition, for functions on the group, appeared for the first time in Arthur's article on the local trace formula,(cf.[A], top of page 58).

In order to avoid to use the fact, unknown in general, that they are only finitely many relative discrete series having a non zero vector fixed by a given compact open subgroup of G, we use Bernstein's center.

We show (cf. Proposition 5) that Eisenstein integrals for proper parabolic subgroups lie in $\mathcal{A}_{temp,c}(H\backslash G)$.

2 The map H_G and the real functions Θ_G, $\|.\|$ and N_d on $H\backslash G$

2.1 Notations

If E is a vector space, E' will denote its dual. If E is real, $E_{\mathbb{C}}$ will denote its complexification.

If G is a group, $g \in G$ and X is a subset of G, $g.X$ will denote gXg^{-1}. If J is a subgroup of G, $g \in G$ and (π, V) is a representation of J, V^J will denote the space of invariant elements of V under J and $(g\pi, gV)$ will denote the representation of $g.J$ on $gV := V$ defined by:

$$(g\pi)(gxg^{-1}) := \pi(x), x \in J.$$

We will denote by (π', V') the contragredient representation of G in the algebraic dual vector space V' of V.

If V is a vector space of vector valued functions on G which is invariant by right (resp., left) translations, we will denote by ρ (resp., λ) the right (resp., left) regular representation of G in V.

If G is locally compact, $d_l g$ will denote a left invariant Haar measure on G and δ_G will denote the modulus function.

Let $\mathbf{F}$ be a non archimedean local field with finite residue field $\mathbf{F}_q$. Unless specified we assume:

$$\text{The characteristic of } \mathbf{F} \text{ is different from 2.} \tag{2.1}$$

Let $|.|_{\mathbf{F}}$ be the normalized absolute value of $\mathbf{F}$.

We will use conventions like in [W]. One considers various algebraic groups defined over $\mathbf{F}$, and a sentence like:

" let A be a split torus " will mean " let A be the group of $\mathbf{F}$-points of a torus, $\underline{A}$, defined and split over $\mathbf{F}$ ". (2.2)

With these conventions, let G be a connected reductive linear algebraic group. Let $\tilde{A}_G$ be the maximal split torus of the center of G. The change with standard notations will become clear later.

Let A be a split torus of G. Let $X_*(A)$ be the group of one-parameter subgroups of A. This is a free abelian group of finite type. Such a group will be called a lattice. One fixes a uniformizer ϖ of $\mathbf{F}$. One denotes by $\Lambda(A)$ the image of $X_*(A)$ in A by the morphism of groups $\underline{\lambda} \mapsto \underline{\lambda}(\varpi)$. By this morphism $\Lambda(A)$ is isomorphic to $X_*(A)$.

If J is an algebraic group, one denotes by $\mathrm{Rat(J)}$ the group of its rational characters defined over $\mathbf{F}$. Let us define:

$$\tilde{\mathfrak{a}}_G = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Rat(G)}, \mathbb{R}).$$

The restriction of rational characters from G to $\tilde{A}_G$ induces an isomorphism:

$$\mathrm{Rat(G)} \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathrm{Rat(\tilde{A}_G)} \otimes_{\mathbb{Z}} \mathbb{R}. \tag{2.3}$$

Notice that $\mathrm{Rat(\tilde{A}_G)}$ appears as a generating lattice in the dual space $\tilde{\mathfrak{a}}'_G$ of $\tilde{\mathfrak{a}}_G$ and:

$$\tilde{\mathfrak{a}}'_G \simeq \mathrm{Rat(G)} \otimes_{\mathbb{Z}} \mathbb{R}. \tag{2.4}$$

One has the canonical map $\tilde{H}_G : G \to \tilde{\mathfrak{a}}_G$ which is defined by:

$$e^{\langle \tilde{H}_G(x), \chi\rangle} = |\chi(x)|_{\mathbf{F}},\ x \in G, \chi \in \mathrm{Rat(G)}. \tag{2.5}$$

The kernel of $\tilde{H}_G$, which is denoted by $\tilde{G}^1$, is the intersection of the kernels of the characters of G, $|\chi|_{\mathbf{F}}$, $\chi \in \mathrm{Rat(G)}$. One defines $X(G) = \mathrm{Hom(G/\tilde{G}^1, \mathbb{C}^*)}$, which is the group of unramified characters of G. One will use similar notations for Levi subgroups of G.

One denotes by $\tilde{\mathfrak{a}}_{G,\mathbf{F}}$, resp., $\tilde{\tilde{\mathfrak{a}}}_{G,\mathbf{F}}$ the image of G, resp., $\tilde{A}_G$, by $\tilde{H}_G$. Then $G/\tilde{G}^1$ is isomorphic to the lattice $\tilde{\mathfrak{a}}_{G,\mathbf{F}}$.

There is a surjective map:

$$(\tilde{\mathfrak{a}}'_G)_{\mathbb{C}} \to X(G) \to 1 \tag{2.6}$$

denoted by $\nu \mapsto \chi_\nu$ which associates to $\chi \otimes s$, with $\chi \in \mathrm{Rat(G)}$, $s \in \mathbb{C}$, the character $g \mapsto |\chi(g)|^s_{\mathbf{F}}$ (cf. [W], I.1.(1)). In other words:

$$\chi_\nu(g) = e^{\langle \nu, \tilde{H}_G(g)\rangle}, g \in G, \nu \in (\tilde{\mathfrak{a}}'_G)_{\mathbb{C}}. \tag{2.7}$$

The kernel is a lattice and it defines a structure of a complex algebraic variety on $X(G)$ of dimension $dim_{\mathbb{R}}\tilde{\mathfrak{a}}_G$. Moreover $X(G)$ is an abelian complex Lie group whose Lie algebra is equal to $(\tilde{\mathfrak{a}}'_G)_{\mathbb{C}}$.

If χ is an element of $X(G)$, let ν be an element of $\tilde{\mathfrak{a}}'_{G,\mathbb{C}}$ such that $\chi_\nu = \chi$. The real part $\mathrm{Re}\,\nu \in \tilde{\mathfrak{a}}'_{\mathrm{G}}$ is independent from the choice of ν. We will denote it by $\mathrm{Re}\,\chi$. If $\chi \in \mathrm{Hom(G, \mathbb{C}^*)}$ is continuous, the character of G, $|\chi|$, is an element of $X(G)$. One sets $\mathrm{Re}\,\chi = \mathrm{Re}\,|\chi|$. Similarly, if $\chi \in \mathrm{Hom(\tilde{A}_G, \mathbb{C}^*)}$ is continuous, the character $|\chi|$ of $\tilde{A}_G$ extends uniquely to an element of $X(G)$ with values in $\mathbb{R}^{*+}$, that we will denote again by $|\chi|$ and one sets $\mathrm{Re}\,\chi = \mathrm{Re}\,|\chi|$.

If P is a parabolic subgroup of G with Levi subgroup M, we keep the same notations with M instead of G.
The inclusions $\tilde{A}_G \subset \tilde{A}_M \subset M \subset G$ determine a surjective morphism $\tilde{\mathfrak{a}}_{M,\mathbf{F}} \to \tilde{\mathfrak{a}}_{G,\mathbf{F}}$, resp., an injective morphism, $\tilde{\tilde{\mathfrak{a}}}_{G,\mathbf{F}} \to \tilde{\tilde{\mathfrak{a}}}_{M,\mathbf{F}}$, which extends uniquely to a surjective linear map between $\tilde{\mathfrak{a}}_M$ and $\tilde{\mathfrak{a}}_G$, resp., injective map, between $\tilde{\mathfrak{a}}_G$ and $\tilde{\mathfrak{a}}_M$. The second map allows to identify $\tilde{\mathfrak{a}}_G$ with a subspace of $\tilde{\mathfrak{a}}_M$ and the kernel of the first one, $\tilde{\mathfrak{a}}_M^G$, satisfies:

$$\tilde{\mathfrak{a}}_M = \tilde{\mathfrak{a}}_M^G \oplus \tilde{\mathfrak{a}}_G. \tag{2.8}$$

If an unramified character of G is trivial on M, it is trivial on any maximal compact subgroup of G and on the unipotent radical of P, hence on G. It allows to identify $X(G)$ to a subgroup of $X(M)$. Then $X(G)$ is the analytic subgroup of $X(M)$ with Lie algebra $(\tilde{\mathfrak{a}}'_G)_{\mathbb{C}} \subset (\tilde{\mathfrak{a}}'_M)_{\mathbb{C}}$. This follows easily from (2.7) and (2.8).One has (cf. [D4], (4.5)),

The map $\Lambda(\tilde{A}_G) \to G/\tilde{G}^1$ is injective and allows to identify $\Lambda(\tilde{A}_G)$ to the sugroup $H_G(\tilde{A}_G)$ of $\tilde{\mathfrak{a}}_G$. (2.9)

Let $\underline{G}$ be the algebraic group defined over $\mathbf{F}$ whose group of $\mathbf{F}$-points is G. Let σ be a rational involution of $\underline{G}$ defined over $\mathbf{F}$. Let H be the group of $\mathbf{F}$-points of an open $\mathbf{F}$-subgroup of the fixed point set of σ. We will also denote by σ the restriction of σ to G.
A split torus of G, A, is said σ-split if A is contained in the set of elements of G which are antiinvariant by σ. Now we explain the change to standard notations: A_G will denote the maximal σ-split torus of the center of G.
Let $\tilde{A}$ be a σ-invariant split torus of G. The involution σ induces an involution, denoted in the same way, on $\tilde{\mathfrak{a}} := \tilde{\mathfrak{a}}_{\tilde{A}}$. Let $\tilde{A}^\sigma$ (resp., $\tilde{A}_\sigma$) be the maximal split torus in the group of elements of $\tilde{A}$ which are invariant (resp., antiinvariant) by σ. Then $\tilde{\mathfrak{a}}^\sigma$ (resp., $\tilde{\mathfrak{a}}_\sigma$) identifies with the set of invariant (resp., antiinvariant) of $\tilde{\mathfrak{a}}$ by σ and $\tilde{A}_\sigma$ is the maximal σ-split torus of $\tilde{A}$.
In particular, one has $A_G = (\tilde{A}_G)_\sigma$ and $\tilde{\mathfrak{a}}_G = \tilde{\mathfrak{a}}_G^\sigma \oplus \mathfrak{a}_G$ where $\tilde{\mathfrak{a}}_G^\sigma$ (resp., $\mathfrak{a}_G$) is the space of invariant (resp., antiinvariant) elements of $\tilde{\mathfrak{a}}_G$ by σ.
We define a morphism of groups $H_G : G \to \mathfrak{a}_G$ which is the composition of $\tilde{H}_G$ with the projection on $\mathfrak{a}_G$ parallel to $\tilde{\mathfrak{a}}_G^\sigma$. We remark that, as it seen easily, $\tilde{H}_G$ commutes to σ. Hence $H_G(H)$ is reduced to zero.
We denote by G^1 the kernel of H_G, which contains H. It contains also $\tilde{G}^1$, hence it is open in G. We denote by $\mathfrak{a}_{G,\mathbf{F}}$ the image of H_G. Let $X(G)_\sigma$ be the connected component of the group of antiinvariant elements of $X(G)$.Then $X(G)_\sigma$ is the analytic subgroup of $X(G)$ with Lie algebra $(\mathfrak{a}'_G)_{\mathbb{C}} \subset (\tilde{\mathfrak{a}}'_G)_{\mathbb{C}}$. The elements of $X(G)_\sigma$ are precisely the characters of G of the form

$$\chi_\nu(g) = e^{\langle \nu, H_G(g) \rangle}, \nu \in (\mathfrak{a}'_G)_{\mathbb{C}}, g \in G.$$

They are exactly the characters of the lattice $G^1 \backslash G$. The group $X(G)_\sigma$ has a natural structure of complex algebraic group. We denote by $X(G)_{\sigma,u}$ the group of unitary

elements of $X(G)_\sigma$.
One has

$$\text{The group } \Lambda(A_G) \text{ identifies by } H_G \text{ to } H_G(A_G). \tag{2.10}$$

Let $\tilde{A}$ be a maximal split torus of G. Let M be the centralizer of $\tilde{A}$ in G. Let us show the following assertion.

$\tilde{H}_M(\tilde{A})$ contains a multiple by $k \in \mathbb{R}^{+*}$ of the coweight lattice of the root system $\Sigma \subset (\tilde{\mathfrak{a}}^G)'$ of $\tilde{A}$ in the Lie algebra of G. Here the coweight lattice is the dual lattice in $\tilde{\mathfrak{a}}^G$ of the root lattice. (2.11)

It is clear that it suffices to prove the assertion for one maximal split torus. Let $\tilde{A}'$ be a maximal split torus of the derived group, G', of G. Let $\underline{\tilde{A}} = \underline{\tilde{A}'}\underline{\tilde{A}_G}$. This is a maximal **F**-split torus of $\underline{G}$ for reasons of dimension. The intersection F of $\underline{\tilde{A}'}$ and $\underline{\tilde{A}_G}$ is finite. Hence one has the exact sequence

$$0 \to F \to \underline{\tilde{A}'} \times \underline{\tilde{A}_G} \to \underline{\tilde{A}} \to 0.$$

Going to **F**-points, the long exact sequence in cohomology implies that $\tilde{A}\tilde{A}'_G$ is of finite index in $\tilde{A}$. Hence the image of $\tilde{A}'\tilde{A}_G$ by $\tilde{H}_M$ is of finite index in the image of $\tilde{A}$. The image of $\tilde{A}'$ (resp., $\tilde{A}_G$) in $\tilde{\mathfrak{a}}$ by $\tilde{H}_M$ is contained in $\tilde{\mathfrak{a}}^G$ (resp., $\tilde{\mathfrak{a}}_G$) and is a lattice Λ_1 (resp., Λ_G) generating $\tilde{\mathfrak{a}}^G$ because $\Lambda_1 + \Lambda_G$ is of finite index in $\Lambda = \tilde{H}_M(\tilde{A})$ which generates $\tilde{\mathfrak{a}}$. Hence the rank of Λ_1 is equal to the dimension of $\tilde{\mathfrak{a}}^G$. The values of the normalized absolute value of **F** are of the form $q^n, n \in \mathbb{Z}$. From the definition of $\tilde{H}_M$, one sees that Λ_1 is included in $(\log q)\Lambda_2$ where $\Lambda_2 \subset \tilde{\mathfrak{a}}^G$ is the coweight lattice of Σ. Both are lattices of the same rank, for reasons of dimension. Our claim follows from the following assertion:

Let $\Lambda_1 \subset \Lambda_2$ be two lattices of the same rank. Then there exists $n \in \mathbb{N}^*$ such that $n\Lambda_2 \in \Lambda_1$, (2.12)

which follows by inverting the matrix, with integers entries, expressing a basis of Λ_1 in a basis of Λ_2.
Let A be a maximal σ-split torus of G and let $\tilde{A}$ be a σ-stable maximal split torus of G which contains A. The roots of A in the Lie algebra of G form a root system (cf. [HW], Proposition 5.9). Let M be the centralizer in G of A, which is σ-invariant. One has $A = A_M$. One deduces like (2.11) that:

$\Lambda(A) \subset \mathfrak{a}$ contains a multiple by $k \in \mathbb{R}^{+*}$ of the coweight lattice of the root system of A in the Lie algebra of G. (2.13)

A parabolic subgroup of G, P, is called a σ-parabolic subgroup if P and $\sigma(P)$ are opposite parabolic subgroups. Then $M := P \cap \sigma(P)$ is the σ-stable Levi subgroup of P. If P is such a parabolic subgroup, P^- will denote $\sigma(P)$.
The sentence : "Let $P = MU$ be a parabolic subgroup of G" will mean that U is the unipotent radical of P and M a Levi subgroup of G. If moreover P is a σ-parabolic subgroup of G, M will denote its σ-stable Levi subgroup.

Let $P = MU$ be a σ-parabolic subgroup of G. Recall that A_M is the maximal σ-split torus of the center of M.
Let A_P^-, (resp., A_P^{--}) be the set of P antidominant (resp., strictly antidominant) elements in A_M. More precisely, if $\Sigma(P)$ is the set of roots of A_M in the Lie algebra of P, and $\Delta(P)$ is the set of simple roots, one has:

$$A_P^- \; (resp., A_P^{--}) = \{a \in A_M | |\alpha(a)|_{\mathbf{F}} \leq 1, \;\; (resp., < 1) \;\; \alpha \in \Delta(P)\}.$$

We define similarly A_P^+ and A_P^{++} by reversing the inequalities. One defines also for $\varepsilon > 0$:

$$A_P^-(\varepsilon) = \{a \in A_M | |\alpha(a)|_{\mathbf{F}} \leq \varepsilon, \;\; \alpha \in \Delta(P)\}.$$

2.2 Some functions on $H\backslash G$

Let $\tilde{A}_0$ be a σ-stable maximal split torus of G, which contains a maximal σ-split torus, A_0, of G. Let P_0 be a minimal parabolic subgroup of G which contains $\tilde{A}_0$. Let K_0 be the fixator of a special point in the apartment of $\tilde{A}_0$ in the Bruhat-Tits building of G. We fix an algebraic embedding

$$\tau : G \to GL_n(\mathbf{F}). \tag{2.14}$$

We may and we will assume that $\tau(K_0) \subset GL_n(\mathcal{O})$ where $\mathcal{O}$ is the ring of integers of $\mathbf{F}$ ([W] I.1)). For $g \in G$, we write:

$$\tau(g) = (a_{i,j})_{i,j=1,..,n}, \; \tau(g^{-1}) = (b_{i,j})_{i,j=1,..,n}.$$

We set

$$\|g\| = sup_{i,j} sup(|a_{i,j}|_{\mathbf{F}}, |b_{i,j}|_{\mathbf{F}}). \tag{2.15}$$

We have (cf. [W] I.1) :

$$\begin{aligned} &\|g\| \geq 1 \text{ for } g \in G, \quad \|g_1 g_2\| \leq \|g_1\| \|g_2\| \text{ for } g_1, \; g_2 \in G \text{ and} \\ &\|k_1 g k_2\| = \|g\| \text{ for } k_1, k_2 \in K_0, \; g \in G. \end{aligned} \tag{2.16}$$

We denote by $(\varepsilon_{M_0}, \mathbb{C})$ the trivial representation of the centralizer M_0 of $\tilde{A}_0$ in G, $(\pi_0, V_0) = (ind_{P_0}^G \varepsilon_{M_0}, ind_{P_0}^G \mathbb{C})$. Let e_0 be the unique element of V_0 invariant by K_0 and such that $e_0(1) = 1$.
We remark that $(\check{\pi}_0, \check{V}_0)$ is isomorphic to (π_0, V_0). For $g \in G$, we set :

$$\Xi_G(g) = \langle \pi_0(g) e_0, e_0 \rangle.$$

The function Ξ_G is biinvariant by K_0.

We will say that two functions f_1 and f_2 defined on a set E with values in $\mathbb{R}_+$ are equivalent on a subset E' of E (we write $f_1(x) \asymp f_2(x), \; x \in E'$), if there exist $C, C' > 0$ such that: (2.17)

$$C' f_2(x) \leq f_1(x) \leq C f_2(x), \; x \in E'.$$

We recall (cf. [W], Lemma II.1.2):

There exist $d \in \mathbb{N}$ and for all $g_1,\ g_2 \in G$, a constant $c > 0$ such that

$$\Xi_G(g_1 g g_2) \leq c\Xi_G(g)(1 + log\|g\|)^d, g \in G. \tag{2.18}$$

We set :

$$\|Hg\| := \|\sigma(g^{-1})g)\|,\ g \in G. \tag{2.19}$$

For a compact subset Ω' of G, we deduce from (2.16):

$$\|Hg\omega\| \asymp \|Hg\|,\ \omega \in \Omega',\ g \in G. \tag{2.20}$$

Let us define the functions Θ_G and $N_d,\ d \in \mathbb{Z}$ by

$$\Theta_G(Hg) = (\Xi_G(\sigma(g^{-1})g))^{1/2},\ g \in G. \tag{2.21}$$

and

$$N_d(Hg) = (1 + log\|Hg\|)^d,\ g \in G. \tag{2.22}$$

(2.20) implies (with $N = N_1$):

$$N(Hg\omega) \asymp N(Hg), g \in G, \omega \in \Omega'. \tag{2.23}$$

The next assertion follows from the definitions and (2.18).

There exists $d \in \mathbb{N}$, and for all $g_1 \in G$ there exists $c > 0$ such that:

$$\Theta_G(Hgg_1) \leq c\Theta_G(g)N_d(Hg), g \in G. \tag{2.24}$$

It follows from the Cartan decomposition for $H\backslash G$ (cf. [BeOh] Theorem 1.1) that there exist a compact subset of G, Ω and a finite set $\mathcal{P}$ of minimal σ-parabolic subgroups of G such that:

$$H\backslash G = \cup_{P \in \mathcal{P}} HA_P^- \Omega. \tag{2.25}$$

Let $P = MU$ be a minimal σ-parabolic subgroup of G and let Ω' be a compact subset of G. We choose a norm on $\mathfrak{a}_M$. By ([L], Lemma 7), we have:

(i) There exist $c, c', C, C' > 0$ such that:

$$Ce^{c\|H_M(a)\|} \leq \|Ha\omega\| \leq C'e^{c'\|H_M(a)\|}, \omega \in \Omega',\ a \in A_P^-, \tag{2.26}$$

(ii)

$$N(Ha\omega) \asymp (1 + \|H_M(a)\|), a \in A_M, \omega \in \Omega.$$

One has the following properties ([cf. [L], Proposition 6).

The function Θ_G is right invariant by $K_0 \cap \sigma(K_0)$.

Let $P = MU$ be a minimal σ-parabolic subgroup of G. Let Ω' be a compact subset of G. There exist $C, C' > 0$ and $d, d' \in \mathbb{N}$ such that

$$C\delta_P^{1/2}(a)N_{-d}(Ha) \leq \Theta_G(Hg) \leq C'\delta_P^{1/2}(a)N_{d'}(Ha), g = a\omega, \omega \in \Omega', a \in A_P^-. \tag{2.27}$$

Lemma 1 *Let dx be a non zero G-invariant measure on $H\backslash G$. There exists $d \in \mathbb{N}$ such that:*

$$\int_{H\backslash G} \Theta_G^2(x) N_{-d}(x) dx < \infty$$

Proof :
Let $P = MU \in \mathcal{P}$ and Ω as in (2.25). From (2.27) one deduces that there exist $C' > 0$ and $d' \in \mathbb{N}$ such that :

$$\Theta_G(Ha\omega) \leq C' \delta_P^{1/2}(a) N_{d'}(Ha), a \in A_P^-, \omega \in \Omega.$$

We can choose Ω large enough in order to have

$$A_P^- \Omega \subset \Lambda_P^- \Omega,$$

where Λ_P^- is the set of P-antidominant elements in $\Lambda(A_M)$. It follows from [KT2], Proposition 2.6, that

There exist constants $C_1, C_2 > 0$ such that:

$$C_1 \delta_P^{-1}(\lambda) \leq \mathrm{vol}(H\lambda\Omega) \leq \mathrm{C}_2 \delta_{\mathrm{P}}^{-1}(\lambda), \lambda \in \Lambda_{\mathrm{P}}^-, \tag{2.28}$$

where $\mathrm{vol}(H\lambda\Omega)$ is the volume of the subset $H\lambda\Omega$ of $H\backslash G$.

From (2.26) (ii) one deduces that for $d'' \in \mathbb{N}$ large enough:

$$\sum_{\lambda \in \Lambda(A_M)} N_{-d''}(H\lambda) < \infty.$$

This implies easily the Lemma. □

3 Tempered functions on $H\backslash G$

3.1 On the Cartan decomposition and lattices

Let $P = MU$ be a σ-parabolic subgroup of G. Let $\Sigma(P)$ be the set of roots of A_M in the Lie algebra of U and let $\Delta(P)$ be the set of simple roots. It will be viewed as a subset of $\mathfrak{a}'_M$. Let us denote by ${}^+\overline{\mathfrak{a}}_P{}'$ (resp., ${}^+\mathfrak{a}'_P$) the set of $\chi \in \mathfrak{a}'_M$ of the form:

$$\chi = \sum_{\alpha \in \Delta(P)} x_\alpha \alpha,$$

where $x_\alpha \geq 0$ (resp., $x_\alpha > 0$) for all $\alpha \in \Delta(P)$.
Let us assume that P is a minimal σ-parabolic subgroup of G. If $Q = LV$ is a σ-parabolic subgroup of G such that $P \subset Q$, let Δ^L be the set of elements of $\Delta := \Delta(P)$ which are roots of A_M in the Lie algebra of L. We denote by A_Q^- the intersection $A_L \cap A_P^-$. For $\varepsilon > 0$ we define

$$A_P^-(Q, \varepsilon) := \{a \in A_P^- \,|\, |\alpha(a)|_{\mathbf{F}} \geq \varepsilon, \alpha \in \Delta^L \textit{ and } |\alpha(a)|_{\mathbf{F}} < \varepsilon, \alpha \in \Delta \setminus \Delta^L\}.$$

Let $\mathcal{P}(P)$ be the set of σ-parabolic subgroups of G which contain P. One has a partition of A_P^-:

$$A_P^- = \cup_{Q\in\mathcal{P}(P)} A_P^-(Q,\varepsilon). \tag{3.1}$$

Moreover for any $Q \in \mathcal{P}(P)$ there exists a compact subset of A_M, $\omega_{\varepsilon,Q}$ such that:

$$A_P^-(Q,\varepsilon) \subset A_Q^- \omega_{\varepsilon,Q}, \tag{3.2}$$

either, introducing Λ_Q^- the set of the Q-antidominant elements of $\Lambda(A_L)$, there exists a compact set $\omega'_{\varepsilon,Q} \subset A$ such that :

$$A_P^-(Q,\varepsilon) \subset \Lambda_Q^- \omega'_{\varepsilon,Q}. \tag{3.3}$$

One uses (2.13) and one introduces a multiple by $k \in \mathbb{R}^{+*}$ of the coweight lattice.
Let $\delta_\alpha \in \mathfrak{a}_M, \alpha \in \Delta(P)$, the fundamental coweights.
Then $\Lambda(A_L)$ contains a sublattice Λ'_L of finite index in $\Lambda(A_L)$, which is generated by $\delta'_\alpha := -k\delta_\alpha \in \Lambda_Q^-$, $\alpha \in \Delta \setminus \Delta^L$ and by $\Lambda(A_G)$. Let $\omega_1,\ldots,\omega_p$ be a basis of $\Lambda(A_G)$ and $\omega'_1,\ldots,\omega'_p$ be the dual basis in $\mathfrak{a}'_G$. Let $\Lambda_Q'^-$ be the semigroup generated by the $\delta'_\alpha, \alpha \in \Delta(Q)$ and $\Lambda(A_G)$, i.e. :

$$\Lambda_Q'^- = \{\prod_{\alpha\in\Delta\setminus\Delta^L} (\delta'_\alpha)^{n_\alpha} | n_\alpha \in \mathbb{N}\}\Lambda(A_G).$$

Then we will see that there exists a finite set in Λ'_L, F_Q such that

$$\Lambda_Q^- \subset \Lambda_Q'^- F_Q. \tag{3.4}$$

In fact if $\lambda \in \Lambda_Q^-$, for each $\alpha \in \Delta \setminus \Delta^L$ (resp., $j = 1,\ldots,p$), one defines n_α (resp., n_j) the largest integer such that $\langle\lambda,\alpha\rangle$ (resp., $\langle\lambda,\omega'_j\rangle$) is less or equal to $-kn_\alpha$ (resp n_j). Then $\lambda' = \prod_{\alpha\in\Delta\setminus\Delta^L}(\delta'_\alpha)^{n_\alpha}\prod_{j=1,\ldots p}\omega_j^{n_j}$ is in $\Lambda_Q'^-$. Moreover $\lambda(\lambda')^{-1}$ lies in a bounded subset of $\Lambda(A_Q)$, as λ varies in Λ_Q^-, hence it lies in a finite set F_Q.
Summarizing, one sees that there exists a compact subset $\omega''_{\varepsilon,Q}$ of A such that :

$$A_P^-(Q,\varepsilon) \subset \Lambda_Q'^- \omega''_{\varepsilon,Q}. \tag{3.5}$$

3.2 $\mathcal{A}(H\backslash G)$, $\mathcal{A}_{temp}(H\backslash G)$, $\mathcal{A}_2(H\backslash G)$

The proof of the following Lemma is analogous to the proof of [D3], Lemma 3.

Lemma 2 *Let f be a function on $H\backslash G$ which is right invariant by a compact open subgroup. The following conditions are equivalent:*
(i) The G-module V_f, generated by the right translates $\rho(g)f, g \in G$, is admissible.
(ii) There exist a smooth admissible representation (π,V) of G, an element v of V and ξ an H-fixed linear form on V such that $f = c_{\xi,v}$ where:

$$c_{\xi,v}(Hg) = \langle\xi,\pi(g)v\rangle, g \in G.$$

(iii) The function f is $ZB(G)$-finite, where $ZB(G)$ is the Bernstein's center of G.

We denote by $\mathcal{A}(H\backslash G)$ the vector space of such functions. An element of this space is A_G-finite, hence there exists a finite set Exp(f) of smooth characters of A_G such that

$$f = \sum_{\chi \in \mathrm{Exp(f)}} f_\chi,$$

where the f_χ are non zero and satisfy for some $n \in \mathbb{N}^*$:

$$(\rho(a) - \chi(a))^n f_\chi = 0, a \in A_G.$$

The elements of Exp(f) are called the exponents of f.
If (π, V) is an admissible smooth G-module and ξ is an H-fixed linear form on V, we define similarly the set of exponents of ξ, $\mathrm{Exp}(\xi)$.
The constant term of f along P, f_P, has been defined in [L], Proposition 2. Denoting by P^-, the σ-parabolic subgroup $\sigma(P)$, $j_{P^-}(\xi)$ has been defined in [L], Theorem 1, where it is denoted by $j_P^*(\xi)$. It is an $M \cap H$-invariant linear form on the normalized Jacquet module $j_P(V)$. One denotes by j_P the canonical projection from V to $j_P(V)$. If $f = c_{\xi,v}$, one has the equality:

$$f_P = c_{j_{P^-}(\xi), j_P(v)}. \tag{3.6}$$

Let us recall a property of the constant term (cf. [D4] Proposition 3.7), in which one has to change right H-invariance to left invariance by changing $g \mapsto f(g)$ into $g \mapsto f(g^{-1})$).

Let $P = MU$ be a minimal σ-parabolic subgroup of G and let $Q = LV$ be a σ-parabolic subgroup of G which contains P. Let K be an open compact subgroup of G. Then there exists $\varepsilon > 0$ such that, for any right K-invariant element of $\mathcal{A}(H\backslash G)$, (3.7)

$$f(a) = \delta_Q^{1/2}(a) f_Q(a), a \in A_M^-(Q < \varepsilon),$$

where $A_M^-(Q < \varepsilon) := \{a \in A_P^- | |\alpha(a)|_{\mathbf{F}} < \varepsilon, \alpha \in \Delta(P) \setminus \Delta^L(P)\}$.

One defines

$$f_P^{ind}(g) := (\rho(g)f)_P, g \in G.$$

As the Jacquet module of an admissible representation is admissible, one deduces from (3.6) that the constant term f_P is an element of $\mathcal{A}(M \cap H\backslash M)$. The union of the set of exponents of $f_P^{ind}(g), g \in G$, $\mathrm{Exp_P(f)}$ is finite, as the Jacquet module of the G-module generated by f is of finite length. This set is called the set of exponents of f along P. If ξ is an H-fixed linear form on a smooth G-module of finite length, one defines similarly $\mathrm{Exp_P}(\xi) = \mathrm{Exp}(\mathrm{j_{P^-}}(\xi))$.

One says that an element of $\mathcal{A}(H\backslash G)$ (resp., an H-fixed linear form on a smooth G-module of finite length) is tempered (resp., square integrable) if for every σ-parabolic subgroup of G, P, the real part of the elements of $\mathrm{Exp_P(f)}$ (resp., $\mathrm{Exp_P}(\xi)$) are contained in ${}^+\bar{\mathfrak{a}}_P{}'$ (resp ${}^+\mathfrak{a}'_P$).

We denote by $\mathcal{A}_{temp}(H\backslash G)$ (resp., $\mathcal{A}_2(H\backslash G)$) the subspace of tempered elements (resp., square integrable) of $\mathcal{A}(H\backslash G)$. Obviously one has:

The spaces $\mathcal{A}_2(H\backslash G) \subset \mathcal{A}_{temp}(H\backslash G)$ are G-invariant suspaces of $\mathcal{A}(H\backslash G)$. (3.8)

Lemma 3 *The following conditions are equivalent:*
(i) The function f is an element of $\mathcal{A}_{temp}(H\backslash G)$ (resp., $\mathcal{A}_2(H\backslash G)$).
(ii) There exists a smooth admissible representation (π, V) of G, an element v of V and a tempered (resp., square integrable) H-fixed linear form on V, ξ, such that:

$$f(Hg) = < \xi, \pi(g)v >, g \in G.$$

Proof :

One uses Lemma 2 and (3.6). □

Definition 1 *Let $f \in \mathcal{A}_{temp}(H\backslash G)$ and let P be a σ-parabolic subgroup of G. Let $\mathrm{Exp}_P^w(f)$ (resp., $\mathrm{Exp}_P^+(f)$) be the set of elements, χ, of $\mathrm{Exp}_P(f)$ such that $\mathrm{Re}\chi = 0$ (resp., is different from zero). The weak constant term of f along P, f_P^w, is the sum of the $(f_P)_\chi$ where χ varies in $\mathrm{Exp}_P^w(f)$. We set $f_P^+ = f_P - f_P^w$.*

Lemma 4 *With the notations of the definition, let $P = MU$, $Q = LV$ be two σ-parabolic subgroups of G such that $P \subset Q$. Let $R = P \cap L$. Then one has:*
(i) $f_Q^w \in \mathcal{A}_{temp}(L \cap H\backslash L)$.
(ii)

$$f_P^w = (f_Q^w)_R^w.$$

Proof :

(i) From the definition of f_Q^w and the fact that $f_Q \in \mathcal{A}(L \cap H\backslash L)$, one sees that f_Q^w is also element of $\mathcal{A}(L \cap H\backslash L)$. The set of exponents $\mathrm{Exp}_R(f_Q)$ is the disjoint union of $\mathrm{Exp}_R(f_Q^w)$ and $\mathrm{Exp}_R(f_Q^+)$. From the transitivity of the constant term (cf. [L], Corollary 1 of Theorem 3), one has $\mathrm{Exp}_R(f_Q) \subset \mathrm{Exp}_P(f)$. Hence if $\chi \in \mathrm{Exp}_R(f_Q^w)$, one has $\mathrm{Re}(\chi) \in^+ \overline{\mathfrak{a}}_P'$ and $\mathrm{Re}(\chi)$ restricted to $\mathfrak{a}_L$ is equal to zero. This implies $\mathrm{Re}(\chi) \in^+ \overline{\mathfrak{a}}_R'$. This implies (i).
Let us prove (ii). We have

$$f_Q = f_Q^w + f_Q^+.$$

Then by the transitivity of the constant term, one has:

$$f_P = (f_Q^w)_R + (f_Q^+)_R = (f_Q^w)_R^w + (f_Q^w)_R^+ + (f_Q^+)_R.$$

Looking to exponents, one concludes that

$$f_P^w = (f_Q^w)_R^w, f_P^+ = (f_Q^w)_R^+ + (f_Q^+)_R.$$

□

3.3 Families of type I of tempered functions

Definition 2 *Let X be a complex algebraic torus. We denote by B the algebra of polynomial functions on X. We denote by X_u the maximal compact subgroup of X. A family, parametrized by X_u, (F_x), of elements of $\mathcal{A}_{temp}(H\backslash G)$ is called a family of type I of tempered functions on $H\backslash G$ if:*
a) There exists a compact open subgroup, J of G such that for all $x \in X_u$, F_x is right invariant by J.
b) For all $g \in G$, the map $x \mapsto F_x(Hg)$ is C^∞ on X_u.
c) For every σ-parabolic subgroup, $P = MU$, of G, there exists a finite family $\Xi_P = \{\xi_1, \dots \xi_n\}$, with possible repetitions, of characters of A_M with values in the group of invertible elements of B, $B^\times$, such that:

$$(\rho(a) - (\xi_1(a))(x)) \dots (\rho(a) - (\xi_n(a))(x)).F^{ind}_{x,P}(g) = 0, a \in A_M, g \in G, x \in X_u,$$

and such that for $i = 1, \dots, n$, the real part of $\xi_i(\cdot)(x)$ is independent of $x \in X_u$ and is element of ${}^+\overline{\mathfrak{a}}_P{}'$. In the following we will denote $\xi_{i,x}(a)$ instead of $(\xi_i(a))(x)$.

We will see later (cf. Theorem 2) examples of such families related to Eisenstein integrals.
The following properties are easy consequences of the definitions.

> If F is a family of type I, parametrized by X_u, of tempered functions on $H\backslash G$, the same is true for the family $\rho(g)F$, for every $g \in G$, with the same control of the exponents. (3.9)

Proposition 1 *Let F be a family of type I, parametrized by X_u, of tempered functions on $H\backslash G$. Then, there exist $d \in \mathbb{N}$ and $C > 0$ such that :*

$$|F_x(Hg)| \le C\Theta_G(Hg)N_d(Hg), g \in G, x \in X_u.$$

Proof :
By using the Cartan decomposition (cf. (2.25)) and a finite number of right translates of F, one sees, using (3.9), (2.23) and (2.24), that it is enough to prove for each element $P = MU$ of $\mathcal{P}$, and each family of type I, parametrized by X_u, of tempered functions on $H\backslash G$, an inequality of this type for $a \in A_P^-$. Now, it follows from (3.7) that there exists $\varepsilon > 0$ such that for all $Q \in \mathcal{P}(P)$ and for all $x \in X_u$:

$$F_{x|A_P^-(Q,\varepsilon)} = (\delta_Q)^{1/2}(F_x)_{Q|A_P^-(Q,\varepsilon)}. \tag{3.10}$$

By (3.1),(3.5), we have $A_P^- \subset \cup_{Q\in\mathcal{P}(P)}\Lambda_Q^-\omega''_{\varepsilon,Q}$. Using a finite number of right translates, (2.23) again and the estimate (2.27) of Θ_G, it is enough to prove that there exist $C > 0$ and $d \in \mathbb{N}$ such that:

$$|(F_x)_Q(\lambda)| \le N_d(H\lambda), \lambda \in \Lambda'^-_Q. \tag{3.11}$$

By assumption on the real part of $\xi_{i,x}$, the eigenvalues $\xi_{i,x}(\lambda), \lambda \in \Lambda'^-_Q$ have a modulus less or equal to 1 which does not depend on $x \in X_u$. We will see that (3.11) follows from the following Lemma.

Lemma 5 *Let Λ be a lattice with basis $\lambda_1, \ldots, \lambda_q$. If $\lambda = i_1\lambda_1 + \ldots + i_q\lambda_q$, we set $|\lambda| = |i_1| + \ldots + |i_q|$. Denote by Λ^+ the set of λ such that the i_j are in $\mathbb{N}$. Let $\xi_{1,x}, \ldots \xi_{n,x}, x \in X_u$ be a C^∞ family of characters of Λ such that:*

$$|\xi_{i,x}(\lambda_j)| \leq 1, x \in X_u.$$

Let $(f_x), x \in X_u$ be a C^∞ family of functions on Λ such that

$$(\rho(\lambda) - \xi_{1,x}(\lambda)) \ldots (\rho(\lambda) - \xi_{n,x}(\lambda)) f_x = 0, x \in X_u, \lambda \in \Lambda.$$

There exist $C > 0, d \in \mathbb{N}$ such that:

$$|f_x(\lambda)| \leq C(1 + |\lambda|)^d, x \in X_u, \lambda \in \Lambda.$$

Proof :
If $i = (i_1, \ldots, i_q) \in \mathbb{Z}^q$ we define

$$\lambda^i = i_1\lambda_1 + \ldots i_q\lambda_q.$$

Let $E_{n,q}$ be the space of maps from $\{0, \ldots, n-1\}^q$ to $\mathbb{C}$. We fix a norm on this vector space. To $x \in X$, we associate the element g_x of $E_{n,l}$ defined by : $g_x(i) = f_x(\lambda^i), i \in \{0, \ldots, n-1\}^q$. Then (cf. [D3] before Lemma 14) there exists a representation ξ_x of Λ, depending only on the family characters of Λ, $\xi_{1,x}, \ldots \xi_{n,x}$, and which depends smoothly of $x \in X_u$, such that for $\lambda \in \Lambda$, the eigenvalues of $\xi_x(\lambda)$ are $\xi_{1,x}(\lambda), \ldots \xi_{n,x}(\lambda)$ and

$$f_x(\lambda^i) = ((\xi_x(\lambda)g_x)(0, \ldots, 0), i \in \mathbb{Z}^q.$$

The eigenvalues of $\xi_x(\lambda_1), \ldots, \xi_x(\lambda_l)$ are of modulus less or equal to 1. Moreover, from the smoothness of ξ_x in $x \in X_u$, one sees that the norms of the endomorphisms $\xi_x(\lambda_i)$ are bounded by a constant independent from $x \in X_u$, as well as there inverse.
From [DOp] Lemma 8.1, one sees that, for some $d' \in \mathbb{N}$, the norm of $\xi_x(\lambda^i)$ is bounded by the product of a constant, independent of $x \in X_u$, with $(1 + |i_1|)^{d'} \ldots (1 + |i_q|)^{d'}$ for $i \in \mathbb{Z}^q$. But the latter is bounded by $(1 + |i_1| + \ldots + |i_q|)^d$, with $d = d'q \in \mathbb{N}$. □

End of the proof of the Proposition. From (2.26), we have:

$$N(Ha) \asymp (1 + \|H_M(a)\|), a \in A_M. \tag{3.12}$$

From the equivalence of norms for finite dimensional vector spaces, one sees that:

$$1 + |i_1| + \ldots + |i_l| \asymp N(H\lambda^i), i \in \mathbb{Z}^l. \tag{3.13}$$

Then the Lemma 5 implies easily (3.11). This finishes the proof of the Proposition. □

Lemma 6 *Let $f \in \mathcal{A}(H\backslash G)$. The following conditions are equivalent:*
(i) The function f is an element of $\mathcal{A}_{temp}(H\backslash G)$.
(ii) There exist $C > 0$ and $d \in \mathbb{N}$ such that:

$$|f(x)| \leq C\Theta_G(x)N_d(x), x \in H\backslash G.$$

Proof :

(i) implies (ii) follows from the Proposition applied to X reduced to one point.
One sees that (ii) implies (i) is the analogue of (i) implies (ii) in [W], Proposition III.2.2. It is proved in the same way. $\square$

Lemma 7 *Let $P = MU$ be a minimal σ-parabolic subgroup of G. Then:*
(i) There exists a compact subset of M, ω_M, such that:

$$M = (H \cap M)A_M\omega_M.$$

(ii) Let $M_P^- := \{m \in M| < \alpha, H_M(m) >\leq 0, \alpha \in \Delta(P)\}$. There exists a compact subset of M, ω'_M, such that

$$M_P^- \subset (H \cap M)A_P^-\omega'_M.$$

Proof :

(i) From the properties of M (cf. [HH] Proposition 1.13 and Lemma 1.9), one has $M = Z_G(A_M)$ and the isotropic component of M, L_2, is contained in H. Let $\tilde{A}_M$ be the maximal split torus of the center of M, C, and let L_1 be the anisotropic component of M. From [BoTi] 4.28, the group $\underline{M}$ is the almost product of $\underline{C}$, $\underline{L}_1$ and $\underline{L}_2$. Hence one sees that CL_1L_2 is of finite index in M, by using a long exact sequence in cohomology. Moreover $\tilde{A}_M\backslash C$ is compact. Hence $((C \cap H)A_M)\backslash C$ is compact. Moreover L_1 is compact. Altogether, this proves (i).
(ii) The set $H_M(\omega_M)$ is finite. Using (i), one deduces from this that there exists a sufficiently dominant element a_0 of A_M such that, if $m \in M_P^-$ is equal to $ha\omega$, with $h \in H \cap M, a \in A_M, \omega \in \omega_M$, one has $aa_0 \in A_P^-$. Hence $\omega'_M = \omega_M a_0^{-1}$ satisfies (ii). $\square$

The following Proposition is the analogue of [W] Lemmas VI.2.1, VI.2.3. The proof is essentially similar.

Proposition 2 *Let F be a family of type I, parametrized by X_u, of tempered functions on $H\backslash G$. Let $Q = LV$ be a σ-parabolic subgroup of G. Then one has:*
(i) The family $(F_x)_Q^w, x \in X_u$ is a family of tempered functions on $(L\cap H)\backslash L$ of type I.
(ii) Let $P = MU$ be a minimal σ-parabolic subgroup of G such that $P \subset Q$. Let $\Delta = \Delta(P)$, $\Delta^L = \Delta(P \cap L) \subset \Delta$ and let

$$D^L(\delta) = \{m \in M_P^-| < \alpha, H_M(m) >\leq -\delta\|H_M(m)\|, \alpha \in \Delta \setminus \Delta^L\}.$$

There exist $\varepsilon > 0, C > 0$ such that:

$$|(F_x)_Q^+(Hm)| \leq C\Theta_L((H \cap L)m)e^{-\varepsilon\|H_M(m)\|}, m \in D^L(\delta), x \in X_u.$$

Proof :

(i) Let $a' \in A_L$ such that $|\alpha(a')|_{\mathbf{F}} < 1$ for all α in $\Delta(Q)$. Let Ξ_Q^w be the set of elements ξ of Ξ_Q such that ξ_x is a unitary character of A_L for all $x \in X_u$. We set $\Xi_Q^+ = \Xi_Q \setminus \Xi_Q^w$.

We recall that there might be repetitions in these families. From the theory of the resultant there exist elements of $B[X]$, R, S, such that:

$$R(X) \prod_{\xi \in \Xi_Q^w} (X - \xi(a')) + S(X) \prod_{\xi' \in \Xi_Q^+} (X - \xi'(a')) = b,$$

where

$$b = \prod_{\xi \in \Xi_Q^w, \xi' \in \Xi_Q^+} (\xi(a') - \xi'(a')).$$

We define

$$S_x = S(\rho(a')) \prod_{\xi' \in \Xi_Q^+} (\rho(a') - \xi_x(a')).$$

where ρ denotes the right regular representation on the space of functions on $(L \cap H)\backslash L$. One sees easily that, from the definition of R, S, the definition of the constant term and of Ξ_Q^w:

$$S_x(F_x)_Q = b(x)(F_x)_Q^w, x \in X_u. \tag{3.14}$$

From the properties of a' and the definition of Ξ_Q^+, one sees that $b(x)$ does not vanish for $x \in X_u$ and is C^∞ on X_u. Hence

$$(F_x)_Q^w = b(x)^{-1} S_x (F_x)_Q, x \in X_u.$$

One shows as in [D4], Proposition 3.11 (i), using recursions relations, that for $l \in L$, the map $x \mapsto (F_x)_Q((H \cap L)l)$ is C^∞ on X_u. Hence $(F_x)_Q^w$ is a smooth family of type I of tempered functions on $(L \cap H)\backslash L$ due to Lemma 4 and our hypothesis on F. This proves (i).
(ii) For $C > 0$, let $D^L(\delta)_C$ be the set of elements m of $D^L(\delta)$ such that $\|H_M(m)\| \leq C$. It follows from Lemma 7 (ii) that it is a compact subset of $(M \cap H)\backslash M$. Hence the assertion of (ii) follows for $m \in D^L(\delta)_C$.
One will see:

If $m' \in M$ is such that $Sup\{< \alpha, H_M(m') > \ |\alpha \in \Delta \setminus \Delta^L\} \leq \delta C/4$, $\|H_M(m')\| \leq C/2$ and $m \in D^L(\delta) \setminus D^L(\delta)_C$, then $mm' \in D^L(\delta/2)$. (3.15)

Let $X = H_M(m), Y = H_M(m')$. Then one has:

$$< \alpha, X + Y > \leq -\delta(\|X\| - C/4), \alpha \in \Delta \setminus \Delta^L.$$

One has to check

$$-\|X\| + C/4 \leq -\|X + Y\|/2,$$

which is equivalent to:

$$\|X + Y\| \leq 2\|X\| - C/2.$$

This follows from the triangular inequality for $X + Y$, and the inequalities $\|X\| \geq C$ and $\|Y\| \leq C/2$. This proves our assertion.
The above facts and Lemma 7 allow to use a finite number of translations to show that it is enough to prove (ii) for $m \in D^L(\delta) \cap A_P^-$ and all $\delta > 0$.

One can assume that Q is proper otherwise $(F_x)^+_Q = 0$.
Let us prove that there exist $t > 0, C_1 > 0$ and $d \in N$ such that, if $a \in A^-_P$ satisfies $< \alpha, H_M(a) > \leq -t$ for $\alpha \in \Delta \setminus \Delta^L$, one has:

$$|(F_x)^+_Q(a)| \leq C_1 \Theta_L((L \cap H)a) N_d(Ha), x \in X_u. \tag{3.16}$$

By (3.7), there exists $t > 0$ such that for a satisfying the above hypothesis, one has the equality:

$$(F_x)_Q(a) = \delta_Q(a)^{-1/2} F_x(a).$$

One deduces from Proposition 1, that there exist $C_2 > 0$ and $d_1 \in \mathbb{N}$ such that

$$|(F_x)_Q(a)| \leq C_2 \delta_Q(a)^{-1/2} \Theta_G(Ha) N_{d_1}(Ha).$$

Applying the right inequality of (2.27) to G and the left inequality to L, and (2.26), one gets that there exist $C_3 > 0$ and $d \in \mathbb{N}$ such that:

$$\delta_Q(a)^{-1/2} \Theta_G(Ha) N_{d_1}(Ha) \leq C_3 \Theta_L((L \cap H)a) N_d(Ha).$$

One deduces from this an inequality like (3.16) for $(F_x)_Q$. A similar inequality for $(F_x)^w_Q$ follows from (i) and Proposition 1 with L instead of G. Hence (3.16) follows by difference.
With the notations of the proof of (i), let us define:

$$r_x(X) := \prod_{\xi \in \Xi^+_Q} (X - \xi_x(a')), x \in X_u.$$

By expanding these polynomials, one gets:

$$r_x(X) = \sum_{i=0,\dots,N} r_{i,x} X^{N-i}.$$

For all $\xi \in \Xi^+_Q$, $|\xi_x(a')|$ is independent of $x \in X_u$ and belongs to the interval $]0, 1[$. Changing a' to a suitable power, one can assume that:

$$|r_{i,x}| \leq 2^{-i} N^{-1}, i = 1, \dots, N-1. \tag{3.17}$$

Let us show, with t and d as in (3.16), the following property.

There exists $C_4 > 0$ such that, for all $n \in \mathbb{N}$ and all $a \in A^-_P$ satisfying $< \alpha, H_M(a) > \leq -t$ for $\alpha \in \Delta \setminus \Delta^L$, one has:

$$|(F_x)^+_Q(a(a')^n)| \leq C_4 2^{-n} \Theta_L((L \cap H)a) N_d(Ha), x \in X_u. \tag{3.18}$$

If $N = 0$, this implies that Ξ^+_Q is empty, hence $(F_x)^+_Q = 0$. So one can assume that $N \in \mathbb{N}^*$. Let

$$C_4 = C_1 Sup\{2^n (N_d(H(a')^n) | n = 0, \dots, N-1\}.$$

If $n < N$, (3.18) follows from (3.16) applied to $a(a')^n$, from the definition of C_4, from the equality

$$\Theta_L((L \cap H)la') = \Theta_L((L \cap H)l), l \in L,$$

as $a' \in A_L$, and from the inequality

$$N(Haa') \leq N(Ha)N(Ha'),$$

which follows easily from the definitions (2.19) and (2.22).
Let $n \geq N$ and let us assume that the inequality (3.18) is true for $n' < n$. It follows from the definitions that $r_x(\rho(a'))(F_x)^+_Q = 0$ for all $x \in X_u$, hence one gets:

$$(F_x)^+_Q(a(a')^n) = -\sum_{i=1,\dots,N} r_{i,x}(F_x)^+_Q(a(a')^{n-i}).$$

The inequality (3.18) for the left side of this equality follows from the induction hypothesis and (3.17). The set $\{a \in A_M \cap D^L(\delta) | \, \|H_M(a)\| \leq t\delta^{-1}\}$ is compact. For all $\varepsilon > 0$, one can find $C > 0$ such that the inequality of (ii) is valid for a in this set. Let:

$$D = \{a \in D^L(\delta) \cap A_M | \, \|H_M(a)\| > t\delta^{-1}\}.$$

For $a \in D$, let n be the largest integer which is less or equal to

$$(\delta\|H_M(a)\| - t)(- < \alpha, H_M(a') >)^{-1},$$

when α varies in $\Delta \setminus \Delta^L$. From the definition of D and the choice of a', n is an element of $\mathbb{N}$. From the definition of $D^L(\delta)$, $a(a')^{-n}$ is in A^-_P and satisfies $\langle \alpha, H_M(a(a')^{-n})\rangle \leq -t$ for all $\alpha \in \Delta \setminus \Delta^L$. By applying (3.18) to $a(a')^{-n}$ instead of a and to the integer n, one gets:

$$(F_x)^+_Q(a) \leq C_4 2^{-n}\Theta_L((L \cap H)a(a')^{-n})N_d(Ha(a')^{-n}), x \in X_u.$$

As it was already observed $\Theta_L((L\cap H)a(a')^{-n}) = \Theta_L((L\cap H)a)$. Moreover, from (2.23) and the definition of n, one sees that there exists $C_5 > 0$

$$N_d(Ha(a')^{-n}) \leq C_5(1 + \|H_M(a)\|)^d.$$

Writing that

$$(\delta\|H_M(a)\| - t)(- < \alpha, H_M(a') >)^{-1} \leq n + 1,$$

for some $\alpha \in \Delta \setminus \Delta^L$, one sees that there exist $r > 0$ and $k \in \mathbb{N}$, independent of $a \in D$, such that:

$$r\|H_M(a)\| \leq n + k.$$

From this it follows that:

$$(F_x)^+_Q(Ha) \leq 2^k C_4 C_5 \Theta_L((L \cap H)a)2^{-r\|H_M(a)\|}(1 + \|H_M(a)\|)^d.$$

In order to finish the proof of (ii), it is enough to remark that there exist $C_6 > 0$ and $\varepsilon > 0$ such that for all $x > 0$,

$$2^{-rx+1}(1 + x)^d \leq C_6 e^{-\varepsilon x}.$$

□

4 Wave packets in the Schwartz space

Definition 3 *The Schwartz space $\mathcal{C}(H\backslash G)$ is the space of functions f on $H\backslash G$, which are right invariant by a compact open subgroup of G and such that for any $d \in \mathbb{N}$, there exists a constant $C_d > 0$ such that:*

$$|f(x)| \leq C_d \Theta_G(x)(N_d(x))^{-1}, x \in H\backslash G.$$

The smallest constant C_d is denoted by $p_d(f)$. It defines a seminorm on $\mathcal{C}(H\backslash G)$.

Lemma 8 *If $f \in \mathcal{A}_{temp}(H\backslash G)$ and $f' \in \mathcal{C}(H\backslash G)$ the integral*

$$\int_{H\backslash G} f(x)f'(x)dx$$

converges absolutely.

Proof :
The lemma follows from Lemma 6 and Lemma 1. □

Let M be the σ-stable Levi subgroup of a σ-parabolic subgroup P of G. Let L be the σ-stable Levi subgroup of an another σ-parabolic subgroup, Q of G. Let $(Q|G|P)_\sigma$ be the set of double cosets $Q\backslash G/P$ having a representative w such that $w.A_M \subset A_L$, which implies $w.M \subset L$. Let $W(L|G|M)_\sigma$ be a set of representatives of $(Q\backslash G/P)_\sigma$ with this property. We recall that $X(G)_{\sigma,u}$ has been identified to a subgroup of $X(M)_{\sigma,u}$.

Definition 4 *An M-family of type I' is a family of type I of tempered functions on $H\backslash G$, F, parametrized by $X(M)_{\sigma,u}$ such that, for any $\chi \in X(M)_{\sigma,u}$,*

$$F_\chi(Hga) = \chi(a)F_\chi(Hg), a \in A_G, g \in G, \chi \in X(M)_{\sigma,u}, \tag{4.1}$$

$$F_{\chi\chi'}(Hg) = \chi'(g)F_\chi(Hg), g \in G, \chi' \in X(G)_{\sigma,u}. \tag{4.2}$$

An M-family of tempered functions on $H\backslash G$ of type I' is said to be of type II' if for any Q as above

$$(F_\chi)_Q^{w,ind}(g) = \sum_{w \in W(L|G|M)_\sigma} (F_{Q,w}(g))_{\chi^w}, \chi \in X(M)_{\sigma,u}, g \in G, \tag{4.3}$$

where for all $w \in W(L|G|M)_\sigma$, $F_{Q,w}(g)$ is a $w.M$-family of type I' of tempered functions on $(L \cap H\backslash L)$ and χ^w denotes the unramified character $w\chi$ of $w.M$.
From the definition it follows that if F is of type II' and $g \in G$, $\rho(g)F$ is also of type II'.

We will give examples of such families, derived from Eisenstein integrals (cf. Theorem 2).

Theorem 1 *Let M be the σ-stable Levi subgroup of a σ-parabolic subgroup of G. Let F be an M-family of tempered functions on $H\backslash G$ of type II'. Let ψ be a C^∞ function on $X_u := X(M)_{\sigma,u}$. Let $d\chi$ be the Haar measure of X_u of volume 1. We define*

$$F_\psi(Hg) = \int_{X_u} \psi(\chi) F_\chi(Hg) d\chi, g \in G.$$

(i) F_ψ is an element of $\mathcal{C}(H\backslash G)$.
(ii) For each $d \in \mathbb{N}$, there exists a continuous semi norm q_d on $C^\infty(X_u)$ such that (with the notation of Definition 3):

$$p_d(F_\psi) \leq q_d(\psi), \psi \in C^\infty(X_u).$$

Proof :
This result is obtained by the same arguments as in the proof of Proposition VI.3.1 of [W] taking F_χ instead of the normalized Eisenstein integrals, $\mu(\omega \otimes \chi) E_P^G sp_\chi \psi$. We just clarify the change of notations and the principal arguments for convenience of the reader.
Let $\mathcal{D}^M$ be the set of finite families of invariant differential operators on X_u. If $D \in \mathcal{D}^M$ and $\psi \in C^\infty(X_u)$, we define:

$$q^M(D, \psi) = Sup\{|d\psi(\chi)||d \in D, \chi \in X_u\}.$$

One wants to prove the following assertion.

$$\text{For } d \in \mathbb{N}, \text{ there exists } D \in \mathcal{D}^M \text{ such that:}$$
$$|F_\psi(Hg)\Theta_G^{-1}(Hg)N_d(Hg)| \leq q^M(D, \psi), \psi \in C^\infty(X_u), g \in G. \tag{4.4}$$

Proceeding as in the proof of Proposition 1, using a finite number of right translations, one is reduced to prove a similar statement for $g \in A_{P_1}^-$, where $P_1 = M_1 U_1$ is a minimal σ-parabolic subgroup of G and A_1 is the maximal σ-split torus of the center of M_1. Let $\Delta_1 := \Delta(P_1)$ and $H_1 := H_{M_1}$. If $a \in A_{P_1}^-$, we set $s(a) := \inf\{< \alpha, H_1(a) >, \alpha \in \Delta_1\} \leq 0$. Let $t < 0$. We first prove (4.4) for F_ψ on $A_1^-(t <) := \{a \in A_{P_1}^- | t \leq s(a) \leq 0\}$. One deduces from (4.2) the following equality.

$$F_\psi(Ha) = \int_{X(G)_{\sigma,u}\backslash X_u} F_\chi(Ha)(\int_{X(G)_{\sigma,u}} \psi(\chi\chi')\chi'(a)d\chi')d\chi, a \in A_1.$$

By classical Fourier transform on the compact abelian group $X(G)_{\sigma,u}$, for $d \in \mathbb{N}$ there exists $D \in \mathcal{D}^M$ such that

$$|\int_{X(G)_{\sigma,u}} \psi(\chi\chi')\chi'(a)d\chi'| \leq (1 + \|H_G(a)\|)^{-d} q^M(D, \psi), a \in A_1.$$

Let $H_1^G(a)$ be the projection of $H_1(a)$ on $\mathfrak{a}_1^G$ parallel to $\mathfrak{a}_G$. By (2.26) there exists $C > 0$ such that

$$N(Ha) \leq C(1 + \|H_1(a)\|) \leq C(1 + \|H_G(a)\|)(1 + \|H_1^G(a)\|), a \in A_1.$$

Hence there exists $C_0 > 0$ such that, for $a \in A_1^-(t <)$ one has $N(Ha) \leq C_0(1+\|H_G(a)\|)$. Using Proposition 1, one deduces the inequality (4.4) for F_ψ on $A_1^-(t <)$.
We proceed now by induction on the semi-simple rank of $H\backslash G$ i.e the dimension of $A_G\backslash A_1$. If this rank is equal to zero the result follows from the first part of the proof. Now we assume that this rank is strictly positive.
For $J \subset \Delta_1$, we define $A_1^-(J) := \{a \in A_{P_1}^-; |J = \{\alpha \in \Delta_1| < \alpha, H_1(a) >= s(a)\}$ so that $A_{P_1}^- \subset \cup_{J\subset\Delta_1} A_1^-(J)$. Let $Q = LV \in \mathcal{P}(P_1)$ be such that $J = \Delta_1 - \Delta_1^L$. If $J = \emptyset$ (which corresponds to $Q = G$) then $A_1^-(J) = \emptyset$, from our hypothesis on the semisimple rank of $H\backslash G$. Hence we can assume $Q \neq G$ and it is enough to prove the statement (4.4) on $A_1^-(J, < t) := \{a \in A_1^-(J)|s(a) < t\}$ for some $t < 0$. By (3.10), there exists $t < 0$ such that, for $Q \in \mathcal{P}(P_1)$ and $a \in A_{P_1}^-$ with $s(a) < t$, one has

$$F_\chi(Ha) = \delta_Q(a)^{1/2}(F_\chi)_Q(Ha) = \delta_Q(a)^{1/2}(F_\chi)_Q^w(Ha) + \delta_Q(a)^{1/2}(F_\chi)_Q^+(Ha).$$

We fix such $t < 0$.
By the assumption on the weak constant term of a M-family of tempered functions on $H\backslash G$ of type II' and by the induction hypothesis, one has the estimate (4.4) with L instead of G for $\int_{X_u} \psi(\chi)(F_\chi)_Q^w(L \cap Ha)d\chi$ on $A_1^-(J, < t)$. By (2.26) and (2.27), there exist $C_0 > 0$ and $r \in \mathbb{N}$ such that

$$\delta_Q^{1/2}(a)\Theta_L(L \cap Ha) \leq C_0 N_r(Ha)\Theta_G(Ha), a \in A_1^-(J, < t). \tag{4.5}$$

This gives the result (4.4) for $\delta_Q(a)^{1/2} \int_{X_u} \psi(\chi)(F_\chi)_Q^w(a)d\chi$ on $A_1^-(J, < t)$.
With the notations of Proposition 2, there exists $\delta > 0$ such that $A_1^-(J, < t) \subset A_G(D^L(\delta) \cap A_{P_1}^-)$. For $a \in A_G$ and $m \in D^L(\delta) \cap A_{P_1}^-$ one has $|(F_\chi)_Q^+(am)| = |(F_\chi)_Q^+(m)|$, $\delta_Q(am) = \delta_Q(m)$ and $\Theta_G(Ham) = \Theta_G(Hm)$. By classical Fourier analysis again, Proposition 2 and (4.5), for $d \in \mathbb{N}$, there exist $r \in \mathbb{N}$ and $D \in \mathcal{D}^M$ such that

$$\delta_Q^{1/2}(am)|\int_{X_u} \psi(\chi)(F_\chi)_Q^+(am)d\chi| \leq$$

$$q^M(D, \psi)(1 + \|H_G(am)\|)^{-d}\Theta_G(Ham)e^{-\epsilon\|H_1(m)\|}N_r(Hm),$$

for $a \in A_G$ and $m \in D^L(\delta) \cap A_{P_1}^-$.
Recall there exists $C > 0$ such that

$$N(Ham) \leq C(1 + \|H_1(am)\|) \leq C(1 + \|H_G(am)\|)(1 + \|H_1^G(m)\|), a \in A_G, m \in A_1.$$

As $(1 + \|H_1^G(m)\|)^d N_r(Hm)e^{-\epsilon\|H_1(m)\|}$ is bounded on A_1, one deduces the result (4.4) for $\delta_Q(a)^{1/2} \int_{X_u} \psi(\chi)(F_\chi)_Q^+(a)d\chi$ on $A_1^-(J, < t)$. This achieves the proof. □

5 A decomposition of $\mathcal{A}_{temp}(H\backslash G)$

5.1 Bernstein's center

We denote by $ZB(G)$ the Bernstein center of G. If (π, V) is a smooth G-module and $z \in ZB(G)$, z acts on V by an endomorphism that we will denote by $\pi(z)$. If V is non

reduced to zero and $ZB(G)$ acts on V by a character ψ, ψ will be called the infinitesimal character of π. Let us denote by (π^*, V^*) the smooth G-module of smooth antilinear forms on V.
Then one shows like in [D3], Equation (1.11), that for $z \in ZB(G)$, there exists a unique z^* in $ZB(G)$ such that for all smooth G-module (π, V), and $v \in V, v^* \in V^*$, one has:

$$(v^*, \pi(z)v) = (\pi^*(z^*)v^*, v).$$

Then one has, for any infinetesimal character ψ:

$$\psi(z^*) = \overline{\psi(z)}. \tag{5.1}$$

If (π, V) is a smooth G-module and ψ is an element of the set $Z\hat{B}(G)$ of characters of $ZB(G)$, we define:

$$V_\psi = \{v \in V | \, for\ some\ n \in \mathbb{N}^*, (\pi(z) - \psi(z))^n v = 0, z \in ZB(G)\}.$$

If V is of finite length, one has:

$$V = \oplus_{\psi \in Z\hat{B}(G)} V_\psi. \tag{5.2}$$

5.2 The case $A_G = \{1\}$

We assume in this subsection that A_G is trivial.
From [KT], Theorem 4.7, one deduces:

> An element f of $\mathcal{A}(H\backslash G)$ is element of $\mathcal{A}_2(H\backslash G)$ if and only if it is an element of $L^2(H\backslash G)$. (5.3)

Using (5.2), one has the following decomposition.

$$\mathcal{A}_{temp}(H\backslash G) = \oplus_{\psi \in Z\hat{B}(G)} \mathcal{A}_{temp}(H\backslash G)_\psi.$$

One has a similar decomposition for $\mathcal{A}_2(H\backslash G)$. An element of $\mathcal{A}_2(H\backslash G)$ generates an admissible unitary subrepresentation of $L^2(H\backslash G)$, which is a finite orthogonal sum of discrete series for $H\backslash G$. Let us prove

$$\mathcal{A}_2(H\backslash G) \subset \mathcal{C}(H\backslash G). \tag{5.4}$$

One proceeds as in the proof of Proposition 1 with X reduces to a single point. One has to replace Lemma 5 by the following property, which follows from [DOp], Corollary 8.2 (ii).

> Let A be an endomorpism of a finite dimensional normed vector space whose eigenvalues are of modulus strictly less than 1. Then for any d in $\mathbb{N}$, there exists a constant $C > 0$ such that: (5.5)
>
> $$\|A^n\| \leq C(1+n)^{-d}, n \in \mathbb{N}.$$

This achieves to prove (5.4).
From Lemma 8, it follows that if $f \in \mathcal{A}_2(H\backslash G)$ and $f' \in \mathcal{A}_{temp}(H\backslash G)$, the integral:

$$\int_{H\backslash G} f(x)\overline{f'}(x)dx$$

converges absolutely. We denote it by (f, f').
Let us show:

$$\mathcal{A}_2(H\backslash G)_\psi \text{ is the direct sum of finetely many irreducible } G\text{-modules.} \tag{5.6}$$

As $\mathcal{A}_2(H\backslash G)$ is endowed with the invariant L^2-scalar product and as there are only finetely many irreducible representations with infinitesimal character equal to ψ, it is enough to prove that for any irreducible smooth G-module, V, the space $\mathrm{Hom_G(V, \mathcal{A}_2(H\backslash G))}$ is finite dimensional. Let us prove that:

$$\text{The space } \mathrm{Hom_G(V, \mathcal{A}(H\backslash G))} \text{ is isomorphic to } V'^H. \tag{5.7}$$

We define a linear map from the first space to the second, $T \mapsto \xi_T$, by defining:

$$\xi_T(v) = T(v)(1), v \in V.$$

The inverse map is given by the coefficient map. Then our claim on $\mathrm{Hom_G(V, \mathcal{A}_2(H\backslash G))}$ follows from the fact that V'^H is finite dimensional (cf. [D4], Theorem 5.7). This finishes the proof of (5.6).
Hence $ZB(G)$ acts by ψ on $\mathcal{A}_2(H\backslash G)_\psi$. As a consequence of (5.1), one sees that if ψ is as above and ψ' is a character of $ZB(G)$ distinct from ψ, then:

$$\mathcal{A}_{temp}(H\backslash G)_{\psi'} \text{ is orthogonal to } \mathcal{A}_2(H\backslash G)_\psi. \tag{5.8}$$

From (5.6), if J is a compact open subgroup of G, the space of J-fixed vectors of $\mathcal{A}_2(H\backslash G)_\psi$ is finite dimensional.

Lemma 9 *(i) Every element f of $\mathcal{A}_{temp}(H\backslash G)$ has a unique decomposition $f = f_2 + f_c$ with $f_2 \in \mathcal{A}_2(H\backslash G)$ and f_c orthogonal to $\mathcal{A}_2(H\backslash G)$.*
(ii) The maps $f \mapsto f_2$, $f \mapsto f_c$ are complementary G-equivariant projectors of $\mathcal{A}_{temp}(H\backslash G)$. We denote by $\mathcal{A}_{temp,c}(H\backslash G)$ the image of the second projector so that one has:

$$\mathcal{A}_{temp}(H\backslash G) = \mathcal{A}_2(H\backslash G) \oplus \mathcal{A}_{temp,c}(H\backslash G).$$

Proof :
(i) The unicity of the decomposition, if it exists, is obvious. By linearity, one can assume, to prove the existence, that $f \in \mathcal{A}_{temp}(H\backslash G)_\psi$ for some character ψ of $ZB(G)$. Let us assume that f is fixed by a compact open subgroup, J, of G. Let f_2 be the orthogonal projection of f on the finite dimensional space $\mathcal{A}_2(H\backslash G)^J_\psi$. Then $f_c = f - f_2$ is orthogonal to $\mathcal{A}_2(H\backslash G)^J_\psi$ and J-invariant. One deduces easily that f_c is orthogonal to $\mathcal{A}_2(H\backslash G)_\psi$, by averaging over J. Then using (5.8) one deduces that f_c is orthogonal to $\mathcal{A}_2(H\backslash G)$. This achieves to prove that f_2 and f_c have the required properties.
(ii) follows from the G-invariance of the space $\mathcal{A}_2(H\backslash G)$ and of the pairing $(.,.)$. $\square$

5.3 General case

We remove the assumption that A_G is trivial. We denote by G' the derived group of G and by H' the intersection $H \cap G'$. As G' is a normal subgroup of G, H acts by conjugacy on H' and $H'\backslash G'$. The action of $h \in H$ on $H'g \in H'\backslash G'$ will be denoted $H'g.h$. Hence H acts linearly on the space of functions on $H'\backslash G'$. Coming back to the definition and considering the representation $h\pi$, one sees that this action preserves $\mathcal{A}(H'\backslash G')$. The action of $h \in H$ on an element f of $\mathcal{A}(H'\backslash G')$ will be denoted $c(h)f$. We may and will choose the function N' such that

$$N'(x) = N(x), x \in H'\backslash G' \subset H\backslash G. \tag{5.9}$$

From the definitions and (2.23) one deduces that for $h \in H$ given, one has:

$$N'(x.h) \asymp N'(x), x \in H'\backslash G'. \tag{5.10}$$

Let A_0 be the maximal split torus of M_0. It contains a maximal split torus, A_0', of $M_0' = M_0 \cap G'$ (cf. section 7). Let K_0' be the fixator of a special point of the apartment of the Bruhat-Tits building of G' corresponding to A_0'. Let $P_0' = P_0 \cap G'$. It is a minimal parabolic subgroup of G'. Let $\Xi_{G'}$ be the corresponding Ξ-function for G'. One deduces from (2.27), applied to G and G', and from the Cartan decomposition (cf. (2.25)) and (2.23) applied to G' that one has:

There exist $C, C' > 0, d, d' \in \mathbb{N}$ such that:

$$CN'_{-d}(H'g)\Theta_G(Hg) \leq \Theta_{G'}(Hg) \leq C'N'_{d'}(H'g)\Theta_G(Hg), g \in G'. \tag{5.11}$$

From this and (2.24) and (2.23), one deduces that, for $h \in H$ given, there exist $C > 0$ and $d \in \mathbb{N}$ such that one has:

$$\Theta_{G'}(x.h) \leq C\Theta_G(x)N'_d(x), x \in H'\backslash G'. \tag{5.12}$$

Together with (5.10), this implies easily that $\mathcal{A}_{temp}(H'\backslash G')$ is preserved by $c(h)$ for all $h \in H$. Moreover, as the conjugacy by $h \in H$ transforms an invariant measure on $H'\backslash G'$ in an invariant measure, and as $c(h)$ preserves $\mathcal{A}(H'\backslash G')$, it preserves also $\mathcal{A}_2(H'\backslash G')$ and $\mathcal{A}_{temp,c}(H'\backslash G')$. It follows that for $f \in \mathcal{A}_{temp}(H'\backslash G')$ and $h \in H$:

$$(c(h)f)_2 = c(h)f_2, (c(h)f)_c = c(h)f_c. \tag{5.13}$$

Let us prove that

The restriction of functions from G to G', $f \mapsto f' := f_{|G'}$ induces a linear map from $\mathcal{A}_{temp}(H\backslash G)$ to $\mathcal{A}_{temp}(H'\backslash G')$. (5.14)

We prove first that this restriction induces a map from $\mathcal{A}(H\backslash G)$ to $\mathcal{A}(H'\backslash G')$. Going back to the definition, it is enough to see that:

A smooth finitely generated admissible G-module is a smooth finitely generated admissible G'-module. (5.15)

As finitely generated admissible smooth G-modules are precisely the smooth G-modules of finite length, this reduces to prove that an irreducible smooth G-module, V, is a smooth G'-module of finite length. Let C be the center of G. The group $G'C$ is of finite index in G and closed in G: this can be seen by a long exact sequence in cohomology. Hence it is open. Moreover, it follows from [BD] Lemma 3.1 (iii), that the product map $G' \times C \to G'C$, is open as a quotient map. From this it follows that there exists a basis of neighbourhood of 1 in G of the form $K'_n C_p$ where K'_n (resp., C_p) is a basis of neighborhood of 1 in G' (resp., C) made of compact open subgroups of G' (resp., C). One may and will assume that the central character of our G-module is trivial on all C_n. From this one deduces the admissibility of the G'-module V. As $G'C$ is of finite index in G, one deduces that the G'-module V is finitely generated. Altogether this proves that V is a smooth G'-module of finite length. This achieves to prove (5.15).
From Lemma 6 and (5.9), (5.12), one sees that the restriction map sends $\mathcal{A}_{temp}(H\backslash G)$ to $\mathcal{A}_{temp}(H'\backslash G')$, which proves (5.14).

One defines 3 linear forms on $\mathcal{A}_{temp}(H\backslash G)$, ξ, ξ_2, ξ_c by:

$$\xi(f) = f(H), \xi_2(f) = f'_2(H'), \xi_c(f) = f'_c(H'), f \in \mathcal{A}_{temp}(H\backslash G).$$

One remarks that if $h \in H$ and $f \in \mathcal{A}_{temp}(H\backslash G)$, one has $(\rho(h)f)' = c(h)f'$. Using (5.13) and the fact H' is normalized by H, one deduces

The linear forms ξ, ξ_2, ξ_c are H-invariant. (5.16)

Lemma 10 *Let f be an element of $\mathcal{A}(H\backslash G)$. The following conditions are equivalent.*
(i) f is an element of $\mathcal{A}_2(H\backslash G)$.
(ii) For all $g \in G$, the restriction of $\rho(g)f$ to G' is an element of $\mathcal{A}_2(H'\backslash G')$.

Proof :
Let $P' = P \cap G'$. This is a σ-parabolic subgroup of G' with $M' = M \cap G'$ as σ-stable Levi subgroup (cf. section 7). Let $A = A_M$, $A' = A_{M'}$. One has $A' \subset A$ (cf (7.9)). Then it is clear that:

$$\{\chi_{|A'} | \chi \in \mathrm{Exp_P(f)}\} = \cup_{\mathrm{g}\in\mathrm{G}} \mathrm{Exp_{P'}}((\rho(\mathrm{g})\mathrm{f})_{|\mathrm{G}'}).$$

Then the Lemma follows from the definitions. $\square$

Proposition 3 *Let $\mathcal{A}_{temp,c}(H\backslash G)$ be the space of elements f of $\mathcal{A}_{temp}(H\backslash G)$ such that for all $g \in G$, the restriction of $\rho(g)f$ to G' is an element of $\mathcal{A}_{temp,c}(H'\backslash G')$.*
If $f \in \mathcal{A}_{temp}(H\backslash G)$ there exists a unique decomposition $f = f_2 + f_c$, with $f_2 \in \mathcal{A}_2(H\backslash G)$ and $f_c \in \mathcal{A}_{temp,c}(H\backslash G)$.
The map $f \mapsto f_2$ (resp., $f \mapsto f_c$) commutes with the right regular representation of G in $\mathcal{A}_{temp}(H\backslash G)$.
In other words, one has a direct sum decomposition of G-modules:

$$\mathcal{A}_{temp}(H\backslash G) = \mathcal{A}_2(H\backslash G) \oplus \mathcal{A}_{temp,c}(H\backslash G).$$

Proof :

From the unicity in Lemma 9, one sees that the decomposition of $f \in \mathcal{A}_{temp}(H\backslash G)$, if it exists, is unique.

Let us prove the existence. For $f \in \mathcal{A}_{temp}(H\backslash G)$ one defines two functions on $H\backslash G$, f_2, f_c, by

$$f_2(Hg) = < \xi_2, \rho(g)f >, f_c(Hg) = < \xi_c, \rho(g)f >, g \in G.$$

In particular $f_{2|G'} = f'_2$. Introducing the finitely generated admissible module generated by f, V_f, one sees from Lemma 2 (ii) that $f_2, f_c \in \mathcal{A}(H\backslash G)$. Moreover the maps $f \mapsto f_2$ and $f \mapsto f_c$ commute with the right regular representation of G. Moreover as $\xi = \xi_2 + \xi_c$, one has $f = f_2 + f_c$.

Let us show that $f_2 \in \mathcal{A}_{temp}(H\backslash G)$. From our definition, for all $g \in G$, $g' \mapsto f(Hg'g)$ is an element of $\mathcal{A}_2(H'\backslash G')$ in particular of $\mathcal{C}(H'\backslash G')$. Let C_{an}be the anisotropic component of the center C of G. Then $G'C_{an}\tilde{A}_G$ is of finite index in G. Let $\tilde{A}^\sigma_G$ be the maximal split torus contained in $\tilde{A}_G \cap H$. The group $\tilde{A}^\sigma_G A_G$ is of finite index in $\tilde{A}_G$. Let Λ_G be the lattice $\Lambda(A_G)$. Using the temperedness criteria (cf. Lemma 6), taking into account that f is smooth and using a finite number of translations, it is enough to prove that for all $f \in \mathcal{A}_{temp}(H\backslash G)$, there exist $C > 0$ and $d \in \mathbb{N}$ such that:

$$|f_2(Hg'\lambda)| \leq C\Theta_G(Hg'\lambda)N_d(Hg'\lambda), g' \in G', \lambda \in \Lambda(A_G). \tag{5.17}$$

One has:

$$\Xi_G(ga) = \Xi_G(g), \Theta_G(Hga) = \Theta_G(Hg), g \in G, a \in A_G.$$

The space V (resp., V_2) of functions on $H'\backslash G'$ generated by the functions (f_λ) (resp., $(f_2)_\lambda$) defined by $H'g' \mapsto f(Hg'\lambda), (resp., f_2(H'g'\lambda)), \lambda \in \Lambda_G$ is finite dimensional as $f, f_2 \in \mathcal{A}(H\backslash G)$. From the temperedness of f, it follows that the eigenvalues of $\rho(\lambda), \lambda \in \Lambda(A_G)$ on V are of modulus 1. As the map $f \to f_2$ is a G-module map, the $\Lambda(A_G)$-module V_2 appears as a quotient of V. Hence the eigenvalues of $\lambda \in \Lambda(A_G)$ acting on V_2 are of modulus 1. From our definitions and (5.4), all elements of V_2 belongs to $\mathcal{C}(H'\backslash G')$. Let d be the dimension of V_2 . We endow V_2 with the norm p_d of $\mathcal{C}(H'\backslash G')$ (cf. Definition 3). Let us denote by F the element $(f_2)_1$ of V_2. From [DOp], Lemma 8.1, and (3.13) , one sees that there exists $C > 0$ such that:

$$p_d((f_2)_\lambda) \leq CN_d(H\lambda)p_d(F), \lambda \in \Lambda_G.$$

Returning to the definition of p_d, this implies:

$$|f_2(Hg'\lambda)| \leq Cp_d(F)\Theta_{G'}(H'g')(N'_d(H'g'))^{-1}N_d(H'\lambda), g' \in G', \lambda \in \Lambda_G. \tag{5.18}$$

From (2.16) and the equality $\|g\| = \|g^{-1}\|$, one deduces

$$N(H\lambda) \leq N(Hg'\lambda)N(Hg').$$

Taking into account (5.9) and (5.12), one sees that (5.17) follows from (5.18). Then one concludes that $f_2 \in \mathcal{A}_{temp}(H\backslash G)$.

Then, as $f_c = f - f_2$, one has $f_c \in \mathcal{A}_{temp}(H\backslash G)$. From the definition of ξ_2 and ξ_c, one has:

$$f_{2|G'} = (f_{|G'})_2, f_{c|G'} = (f_{|G'})_c.$$

As the same is true for the right translates of f by elements of G, one sees that $f_2 \in \mathcal{A}_2(H\backslash G)$ from Lemma 10 and then that $f_c \in \mathcal{A}_{temp,c}(H\backslash G)$. This achieves to prove the Proposition. $\square$

One has seen that $G'C$ is closed in G and of finite index in G. Let us consider the continuous canonical projection:

$$G'C \to G'\backslash G'C \approx (C \cap G')\backslash C.$$

As $C \cap G'$ is finite, if $(g'_n c_n)$ converges in G, where, for all $n \in \mathbb{N}$, $g'_n \in G', c_n \in C$, there exists a subsequence of (c_n) which converges. This implies:

If C_1 is a closed subgroup of C, $G'C_1$ is closed in G. (5.19)

Let A_G^0 be the maximal compact subgroup of A_G and $\tilde{A}_G^\sigma$ the maximal split torus of $\tilde{A}_G \cap H$. This shows that $G^2 = G'C_{an}\tilde{A}_G^\sigma A_G^0$ is closed in G. It is obviously also normal in G.
We recall that G^1 is the kernel of H_G, which contains H and it is open in G. Obviously one has $G^2 \subset G^1$. Let us show:

The group G^2 is of finite index in G^1. (5.20)

We recall that $\mathfrak{a}_{G,\mathbf{F}}$ is the image of H_G. This is a lattice in $\mathfrak{a}_G$. The image of A_G by H_G is a lattice in $\mathfrak{a}_G$ of rank equal the dimension of $\mathfrak{a}_G$. Hence the image of A_G is of finite index in $\mathfrak{a}_{G,\mathbf{F}}$. The group $G^2\Lambda(A_G)$ is of finite index in G as $C_{an}\tilde{A}_G^\sigma A_G$ is of finite index in C and $G'C$ is of finite index in G. Hence $G^2\Lambda(A_G)\backslash G^1\Lambda(A_G)$ is finite. As $G^1 \cap \Lambda(A_G)$ is reduced to 1, the natural map $G^2\backslash G^1 \to G^2\Lambda(A_G)\backslash G^1\Lambda(A_G)$ is injective. This achieves to prove (5.20).
This implies that $G'\tilde{A}_G^\sigma$ is cocompact in G^1. For a suitable normalization of measures one has

$$\int_{H\backslash G^1} f(Hg_1)d\dot{g}_1 = \int_{G'\tilde{A}_G^\sigma\backslash G^1}\int_{H\cap G'\backslash G'} f(Hg'g_1)d\dot{g}'d\dot{g}_1, f \in C_c(H\backslash G^1). \tag{5.21}$$

If $X \in \mathfrak{a}_{G,F}$, one denotes by $(H\backslash G)^X$ the set of $Hg \in H\backslash G$ such that $H_G(g) = X$. Such a set is the right translate of $H\backslash G^1$ by any $g \in G$ such that $H_G(g) = X$. It is is open in $H\backslash G$ and the invariant measure on $H\backslash G$ induces a measure on $(H\backslash G)^X$. If f is a complex valued function on $H\backslash G$, f^X will denote the restriction of f to $(H\backslash G)^X$. If f' is an other function, we denote:

$$(f, f')^X = \int_{(H\backslash G)^X} f(x)\overline{f'(x)}dx,$$

when it is is defined.

Proposition 4 *(i) For all $f \in \mathcal{A}_2(H\backslash G)$ (resp., $f' \in \mathcal{A}_{temp}(H\backslash G)$) and for all $X \in \mathfrak{a}_{G,\mathbf{F}}$, f^X is square integrable (resp., the integral $(f, f')^X$ converges absolutely).*
(ii) For all $f \in \mathcal{A}_{temp}(H\backslash G)$ and for all $X \in \mathfrak{a}_{G,\mathbf{F}}$, $(f_2, f_c)^X = 0$.

Proof :

One reduces, by translation, to $X = 0$. Then (i) and (ii) follows easily from Proposition 3 and from the integral formula (5.21). $\square$

6 Some properties of Eisenstein integrals

6.1 Eisenstein integrals

Let us recall some results of [BD]. Let $P = MU$ be a σ-parabolic subgroup of G, (δ, E) be a finite length smooth representation of M. Let I_δ be the space of the induced representation $i_{K_0\cap P}^{K_0}\delta_{|K_0\cap P}$. Let $i_P^G E_\chi$ or simply I_{δ_χ} be the space of the normalized induced representation $\pi_\chi := i_P^G(\delta_\chi)$, $\chi \in X(M)_\sigma$, where $\delta_\chi = \delta \otimes \chi$. The restriction of functions from G to K_0 determines an isomorphism of K_0-modules between I_{δ_χ} and I_δ. One denotes by $\overline{\pi}_\chi$ the representation of G on I_δ deduced from π_χ by " transport de structure" by this isomorphism.

If $\varphi \in I_\delta$ and $\chi \in X(M)_\sigma$, one denotes by φ_χ the element of the space I_{δ_χ} corresponding to φ by this isomorphism.

Let B be the algebra of polynomial functions on $X(M)_\sigma$, which is generated by the functions $b_m, m \in M$ defined by $b_m(\chi) = \chi(m)$. One has:

$$\text{For all } \varphi \in I_\delta \text{ et } g \in G,\ \chi \mapsto \overline{\pi}_\chi(g)\varphi \text{ is an element of } I_\delta \otimes B. \tag{6.1}$$

Let $\mathcal{O}$ be the union of the open (P, H)-double cosets in G. There exists a set of representatives, $\overline{\mathcal{W}}_M^G$, of these open (P, H)-double cosets which depends only on M and not on P. Moreover for all $x \in \overline{\mathcal{W}}_M^G$, $x^{-1}.P$ is a σ-parabolic subgroup of G (cf [BD] Lemma 2.4). Let A be a maximal σ-split torus of M. We may (cf. [BD], beginning of section 2.4 and Lemma 2.4) and we will assume that for all $x \in \overline{\mathcal{W}}_M^G$, $x^{-1}.A$ is a σ-split torus. One says that x is A-good. Then $x^{-1}.M$ is the σ-stable Levi subgroup of $x^{-1}.P$ (cf [CD], Lemma 2.2). From (7.11), one can even assume that $\overline{\mathcal{W}}_M^G$ is a subset of G'.
One sets $J_\chi = \{\varphi \in I_{\delta_\chi} | \mathrm{Supp}(\varphi) \in \mathcal{O}\}$ and we define:

$$\mathcal{V}_\delta := \oplus_{x\in\overline{\mathcal{W}}_M^G} (E'_\delta)^{M\cap x.H}.$$

Let $\chi \in X(M)_\sigma$. To $\eta \in {E'_\delta}^{M\cap x.H}$, one associates $j(P, \delta_\chi, \eta) \in J'_\chi$ defined by:

$$j(P, \delta_\chi, \eta)(\varphi) = \int_{H\cap(x^{-1}.M)\backslash H} < \varphi(xh), \eta > d\dot{h}, \varphi \in J_\chi. \tag{6.2}$$

Then one has (cf. [BD], Theorem 2.8):

> For χ in an open dense subset, O, of $X(M)_\sigma$, $j(P, \delta_\chi, \eta)$ extends uniquely to an H-invariant linear form on I_{δ_χ}, $\xi(P, \delta_\chi, \eta)$. There exists a non zero polynomial on $X(M)_\sigma$, q, such that for all $\varphi \in I_\delta$, the map $\chi \mapsto q(\chi)\langle\xi(P, \delta_\chi, \eta), \varphi_\chi\rangle$, defined on O, extends to a polynomial function on $X(M)_\sigma$. (6.3)

The Eisenstein integrals are defined, as rational functions of $\chi \in X(M)_\sigma$, by

$$E_P^G(\eta \otimes \varphi_\chi)(Hg) = \langle\xi(P, \delta_\chi, \eta), \pi_\chi(g)\varphi_\chi\rangle, g \in G, \varphi \in I_\delta. \tag{6.4}$$

Let $x \in \overline{\mathcal{W}}_M^G$ and $\eta \in {E'_\delta}^{M\cap x.H}$. Then from our choice , $x^{-1}.P$ is a σ-parabolic subgroup and $x^{-1}.M$ is its σ-stable Levi. One can choose 1 as an element of $\overline{\mathcal{W}}_{x^{-1}.M}^G$ and one has

$E'^{M\cap x.H}_{\delta} = E'^{(x^{-1}.M)\cap H}_{x^{-1}\delta} \subset \mathcal{V}_{x^{-1}.\delta}$. Let $\chi \in X(M)_\sigma$. The map $\varphi \to \lambda(x^{-1})\varphi$ is a bijective intertwining map between $i^G_P(\delta_\chi)$ and $i^G_{x^{-1}.P}((x^{-1}\delta)_{x^{-1}\chi})$. By "transport de structure", one sees

$$E^G_P(\eta \otimes \varphi_\chi) = E^G_{x^{-1}.P}(\eta \otimes (\lambda(x^{-1})\varphi_\chi)), g \in G, \varphi \in I_\delta. \tag{6.5}$$

6.2 Examples of families of type II' of tempered functions

Let $\chi \in X(M)_\sigma, \chi' \in X(G)_\sigma$. We recall that $X(G)_\sigma$ has been identified to a subgroup of $X(M)_\sigma$. One has $\mathrm{Supp}(\varphi_\chi) = \mathrm{Supp}(\varphi)\ P$, $\mathrm{Supp}(\varphi) = \mathrm{Supp}(\varphi_\chi)\cap\ K_0$. Hence φ_χ is in J_χ if and only if $\varphi \in J$, where J is the subspace of elements of I_δ such whose support is contained in $\mathcal{O} \cap K_0$. Moreover:

$$\varphi_{\chi\chi'} = \chi'\varphi_\chi \tag{6.6}$$

and $\pi_{\chi\chi'}$ naturally identifies with $\pi_\chi \otimes \chi'$. As $\chi' \in X(G)_\sigma$, it is trivial on H. As $\overline{\mathcal{W}}^G_M \subset G'$, χ' is trivial on this set. It follows easily from (6.2), that one has:

$$\langle \xi(P, \delta_{\chi\chi'}, \eta), \varphi_{\chi\chi'}\rangle = \langle \xi(P, \delta_\chi, \eta), \varphi_\chi\rangle, \varphi \in J. \tag{6.7}$$

From (6.3), one sees that for $\chi \in O$, the above equality is also valid for all $\varphi \in I_\delta$. From the previous discussion, one deduces the following equality.

$$E^G_P(\eta \otimes \varphi_{\chi\chi'}) = \chi' E^G_P(\eta \otimes \varphi_\chi), \varphi \in I_\delta. \tag{6.8}$$

One will show that one can and will choose q in (6.3) such that:

$$q(\chi\chi') = q(\chi), \chi \in X(M)_\sigma, \chi' \in X(G)_\sigma. \tag{6.9}$$

In fact, from (7.3) it follows that the element λ of Lemma 2.5 in [BD] can be choosen in G'. It follows that the element μ in l.c. belongs to G'. Hence the elements m occuring in [BD], Lemma 2.6 can be taken in G', as it is easily checked from its proof. The same is true for elements of M occuring in l.c. Theorem 2.7 (i). Hence the polynom q of this Theorem satisfies the above property. But it is precisely the polynomial function that we are interested with, by Theorem 2.8 (iv) of [BD].
Let $Q = MV$ be a σ-parabolic subgroup with the same σ-stable Levi subgroup than P. We introduce the intertwining integrals (cf. e.g. [CD], section 4.1). Using (6.6) and the definition of intertwining integrals, on sees:

$$A(Q, P, \delta_{\chi\chi'})\varphi_{\chi\chi'} = \chi'(A(Q, P, \delta_\chi)\varphi_\chi) \tag{6.10}$$

which implies that the restriction to K_0 of both sides are equal.
Deriving intertwining integrals from ξ-functions for the group G viewed as a symmetric space for $G \times G$ as in [BrD] section 4, one sees that

> There exists a non zero polynomial q' on $X(M)_\sigma$, invariant by $X(G)_\sigma$, such that $\chi \mapsto q'(\chi)A(Q, P, \delta_\chi)$ is polynomial in the compact realization. (6.11)

Let $\varphi \in I_\delta$. One introduces the B-matrices as in [CD], Proposition 6.2. Taking into account this equality, (6.6), (6.7) and the definition of B-matrices, one sees:

$$B(P, Q, \delta_{\chi\chi'}) = B(P, Q, \delta_\chi).$$

From l.c., one has:

The map $\chi \mapsto q(\chi)q'(\chi)B(P,Q,\delta_\chi)$ is polynomial on $X(M)_\sigma$. (6.12)

Theorem 2 *Let $\varphi \in I_\delta$ and $\eta \in \mathcal{V}_{\delta,2} = \oplus_{x\in\overline{\mathcal{W}}_M^G} E_2'^{M\cap x.H}$ where $E_2'^{M\cap x.H}$ is the space of square integrable $M \cap x.H$-invariant linear forms on E. There exists a non zero polynomial on $X(M)_\sigma$, p, invariant by $X(G)_\sigma$, such that $\chi \mapsto p(\chi)E_P^G(\eta \otimes \varphi_\chi), \chi \in X(M)_{\sigma,u}$ is a family of type II' of tempered functions on $H\backslash G$.*

Proof :

For $\chi \in X(M)_{\sigma,u}$, the Eisenstein integrals are tempered (cf. [CD], Theorem 7.8). It follows from this and from the properties of the Jacquet modules of induced representations (cf. [D3] Lemma 7, especially Equation (3.4)), that for q as in (6.3), (6.9), the family $q(\chi)E_P^G(\eta \otimes \varphi_\chi), \chi \in X(M)_{\sigma,u}$ is a family of type I' of tempered functions on $H\backslash G$.
Then the Proposition follows from [CD], Theorem 8.4 and from (6.3) applied to Levi subgroups of σ-parabolic subgroups of G instead of G and (6.11),(6.12) applied to C-functions. Notice that in [CD], the hypothesis on the characteristic of the residue field will be removed in a new version. □

Proposition 5 *With the notations of the above Proposition, $E_\chi = p(\chi)E_P^G(\eta \otimes \varphi_\chi)$ is element of $\mathcal{A}_{temp,c}(H\backslash G)$ for all $\chi \in X(M)_{\sigma,u}$.*

Proof :

First, let us assume that $A_G = \{1\}$. If ψ is a character of $ZB(G)$, every $f \in \mathcal{A}_2(H\backslash G)_\psi$ generates an admissible G-submodule of $L^2(G/H)$. It follows easily that the set of ψ such that $\mathcal{A}_2(H\backslash G)_\psi$ is different from zero is at most countable. For reasons of infinitesimal character and (5.8), it follows that the set of χ such that E_χ is not orthogonal to $\mathcal{A}_2(H\backslash G)_\psi$ is at most countable. But if $f \in \mathcal{A}_2(H\backslash G)_\psi$, the map $\chi \mapsto (f, E_\chi)$ is continuous on $X(M)_\sigma$, by Lebesgue dominated convergence and Proposition 1. As it is equal to zero on the complementary of a countable set, it is zero everywhere by density and continuity. This proves the proposition if $A_G = \{1\}$.
Let us return to the general case. From (6.5), one considers only Eisenstein integrals corresponding to the open (P,H)-double coset PH.
Let G' be the group of $\mathbf{F}$-points of the derived group of $\underline{G}$. One will compare the Eisenstein integrals for P and $P' = P \cap G'$. The restriction to M' of δ is unitary and of finite length (cf. (5.15)). From (7.12), one deduces:

The restriction of functions from G to G' induces a bijective map from $i_P^G E$ to $i_{P'}^{G'} E$. (6.13)

Let us show that:

If δ is unitary, this is a unitary operator. (6.14)

This follows from the fact that the unipotent radical, U^-, of $P^- = \sigma(P)$ is contained in G' as the scalar product on $i_P^G E$ is given by:

$$(\varphi, \varphi') = \int_{U^-} (\varphi(u^-), \varphi'(u^-))du^-, \varphi, \varphi' \in i_P^G E.$$

One will compute $\langle j(P, \delta_\chi, \eta), \varphi \rangle$ for $\varphi \in J_\chi$ and $\eta \in E_\delta'^{M\cap H}$, using the integral formula (7.13). With the notations of (7.13), there exist a certain family (x_i) in $\overline{\mathcal{W}}_{M'}^{G'}$ and $m_i \in M$ such that $y_i = m_i x_i \in H$ and $H = \cup_i (M \cap H) y_i H'$. One checks easily that $\delta(m_i^{-1})\eta \in E_\delta'^{M\cap x_i.H}$. Using the equality

$$\varphi(y_i h') = \delta'_\chi(m_i)\varphi(x_i h'),$$

one deduces the equality:

$$\langle j(P, \delta_\chi, \eta), \varphi \rangle = \sum_i \chi(m_i) \langle j(P', \delta_{\chi_{|M'}}, \delta'(m_i^{-1})\eta), \varphi_{|G'} \rangle.$$

From the definition of Eisenstein integrals, one deduces:

$$E_P^G(\delta_\chi, \eta \otimes \varphi_\chi)_{|G'} = \sum_i \chi(m_i) E_{P'}^{G'}(\delta_{\chi|M'}, \delta(m_i)^{-1}\eta \otimes (\varphi_\chi)_{|G'}), \varphi \in I_\delta. \tag{6.15}$$

"Par transport de structure " and a conjugacy by m_i one sees that if η is an $M \cap H$-square integrable form then $\delta'(m_i^{-1})\eta$ is $M \cap x_i.H$-square integrable. Hence it is $M' \cap x_i.H$-square integrable due to Lemma 10. Then the Proposition follows from the first part of the proof and from Proposition 3. □

7 Some properties of the derived group

Recall that we denote by G' the group of $\mathbf{F}$-points of the derived group of $\underline{G}$.
If $\tilde{A}_0$ is maximal split torus of G, there exists a maximal split torus of G', $\tilde{A}'_0$ such that $\underline{\tilde{A}}_0 = \underline{\tilde{A}}_G \underline{\tilde{A}}'_0$: this has been proved for at least one $\tilde{A}_0$ in the proof of (2.11) and the result follows from the fact that all maximal split tori of G are G-conjugate. It is clear that $\tilde{A}'_0$ is the maximal split torus of $\tilde{A}_0 \cap G'$ and one has

The map $\tilde{A}_0 \to \tilde{A}'_0$ is a bijection between the set of maximal split torus of G and the set of maximal split torus of G'. (7.1)

Hence one has:

All maximal split tori of G are G'-conjugate. (7.2)

If $\lambda \in \Lambda(\tilde{A}_0)$, we let P_λ be the parabolic subgroup of G which contains $\tilde{A}_0$, such that the roots of $\tilde{A}_0$ in the Lie algebra of P_λ are the roots α such that $|\alpha(\lambda)|_{\mathbf{F}} \leq 1$.
If P is a parabolic subgroup of G and $\tilde{A}_0 \subset P$, there exists $\lambda \in \Lambda(\tilde{A}_0)$ such that $P = P_\lambda$. One has seen (cf. after (2.11)) that the lattice $\Lambda(\tilde{A}'_0)\Lambda(A_G)$ is of finite index in $\Lambda(\tilde{A}_0)$. Hence a power of λ is element of this lattice, hence of the form $\lambda'\mu$ where $\lambda' \in \Lambda(\tilde{A}'_0)$ and $\mu \in \Lambda(A_G)$. One deduces from the definitions the equality:

$$P_\lambda = P_{\lambda'}. \tag{7.3}$$

Hence one can even choose $\lambda \in \Lambda(\tilde{A}'_0)$. From this and [BD], Equation (2.7), it follows easily that $P \cap G'$ is a parabolic subgroup of G'. Reciprocally if $P' = P_\lambda \cap G'$ for $\lambda \in \Lambda(\tilde{A}'_0)$, looking to Lie algebras, one sees that P_λ is the unique parabolic subgroup of G which contains P'. Altogether we have shown:

The map $P \mapsto P \cap G'$ is a bijection between the sets of parabolic subgroups of G and G'. (7.4)

If P and Q are opposed parabolic subgroups of G, one can choose a maximal split torus $\tilde{A}_0 \subset P \cap Q$ and $\lambda \in \Lambda(\tilde{A}_0)$ such that $P = P_\lambda$ and $Q = P_{\lambda^{-1}}$. As above we can take $\lambda \in \Lambda(\tilde{A}'_0)$. This implies that $P \cap G'$ and $Q \cap G'$ are opposed parabolic subgroups of G'. One shows similarly that if P', Q' are opposed parabolic subgroups of G' and P (resp., Q) is the unique parabolic subgroup of G which contains P' (resp., Q') then P and Q are opposed. It follows easily that the map $P \mapsto P' = P \cap G'$ is a bijection between the sets of σ-parabolic subgroups of G and G', and in particular between the sets of minimal σ-parabolic subgroups. Then it follows:

The map $M \mapsto M \cap G'$ is a bijection for the sets Levi subgroups of σ-parabolic subgroups of G and G', (7.5)

which can be specialized to Levi subgroups of minimimal σ-parabolic subgroups. The map which associates to such a Levi subgroup its unique maximal σ-split torus is a bijection (cf. [HW] Proposition 4.7 and Lemma 4.5). Hence it follows that the correspondence which associates to a maximal σ-split torus of G, A, the maximal split torus, A' of its intersection with G' is a bijection between the sets of maximal σ-split tori of G and G'. Then one has:

The split torus A is the unique maximal σ-split torus such that $A' \subset A$. (7.6)

This implies, for reason of dimensions, that $\underline{A} = \underline{A'}\underline{A_G}$. From which it follows:

Let A_1 be a maximal σ-split torus. If $A'_1 = g'.A'$, for some $g' \in G'$, one has $A_1 = g'.A$. (7.7)

Hence it follows from (7.2) that:

All the maximal σ-split tori of G are G'-conjugate. (7.8)

Let $P = MU$ be a σ-parabolic subgroup of G. Let us show that

$$A_{M'} \subset A_M. \tag{7.9}$$

One has only to check that $A_{M'}$ is in the center of M. But the derived group of $\underline{M}$, $\underline{M}_{der}$ is contained in $\underline{G'}$, hence contained in $\underline{M'}$. As $\underline{M}$ is the almost product of $\underline{M}_{der}$ and its center, an element of $\underline{M}$ which commutes to $\underline{M}_{der}$ is an element of the center. Our claim follows easily.
There exists $\lambda \in \Lambda(A_M)$ such that $P = P_\lambda$. As in the proof of (7.4), one shows that:

There exists $\lambda \in A_{M \cap G'}$ such that $P = P_\lambda$. (7.10)

If A is as above, let us define the quotient $W_G(A)$ of its normalizer in G by its centralizer. From the interpretation of this group as the Weyl group of a certain root system (cf. [HW], Proposition 5.9), one sees that $W_{G'}(A')$ is in bijection with $W_G(A)$. Hence there are representatives of all elements of $W_G(A)$ in the normalizer in G' of A. From this and (7.2), it follows that, with the notations of [BD] (2.14), one may and will choose $\mathcal{W}^G \subset G'$. Hence, from [BD], Lemma 2.4, one may and will choose

$$\overline{\mathcal{W}}^G_M \subset G'. \tag{7.11}$$

Let us prove:

$$\text{If } P = MU \text{ is a parabolic subgroup of } G \text{ one has } PG' = G. \tag{7.12}$$

Consider the canonical map $G' \mapsto P\backslash G$. It factors through $(P \cap G')\backslash G'$ which is compact. Hence the image of G' is compact. On the other hand, let $\overline{U}$ the unipotent radical of the opposite parabolic of P. It is contained in G'. Hence the image of G' contains $P\backslash P\overline{U}$ which is dense in $P\backslash G$. Hence the image of G' is equal to $P\backslash G$. This proves (7.12).
Let P be a σ-parabolic subgroup of G and let $P' = P \cap G$, $H' = H \cap G'$. If $x \in G' \cap (PH)$, $x^{-1}.P$ is a σ-parabolic subgroup of G, hence $G' \cap x^{-1}.P = x^{-1}.P'$ is a σ-parabolic subgroup of G'. Hence $G' \cap (PH)$ is a union of open (P', H')-double cosets, $P'x_iH'$ for a certain family (x_i) of elements of $\overline{\mathcal{W}}^{G'}_{M'}$, where $M' = M \cap G'$. Hence there exists a maximal σ-split torus A' of $M' = M \cap G'$ such that for all $x \in \overline{\mathcal{W}}^{G'}_{M'}$, $x^{-1}.A'$ is a σ-split torus. Let A be a maximal σ-split torus of M containing A'. One has $\underline{A} = \underline{A'A_G}$. This implies that $x^{-1}.A$ is a σ-split torus. Hence, with the terminology of [CD] Definition 2.1, x is A-good.
Let us prove that H is a finite union of double $(H \cap M, H')$ cosets. One has:

$$PH \subset PG' = G, PH \subset P(PH \cap G').$$

Then $H \subset \cup_i Px_iH'$. If $H \cap Px_iH' \neq \emptyset$, there is $p_i \in P$ such that $y_i = p_ix_i \in H$. As $y_i \in H$, $y_i^{-1}.A$ is σ-split hence y_i is A-good. Hence from [CD], Equation (7.1), one sees that $p_i \in M$. We will denote it by m_i. Hence $y_i = m_ix_i \in H$. Then $H \cap (Py_iH') = (P \cap H)y_iH'$. But $(P \cap H) = M \cap H$, as P is a σ-parabolic subgroup of G, so that one has $H \cap Px_iH' = (M \cap H)y_iH'$.
Hence, for a suitable normalization of measures one has:

$$\int_{M\cap H\backslash H} f((H \cap M)h)d\dot{h} = \sum_i \int_{(H'\cap y_i^{-1}M')\backslash H'} f((H \cap M)y_ih')d\dot{h}', f \in C_c(M \cap H\backslash H). \tag{7.13}$$

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